# Strings In and Out of Equilibrium
# &
# Cosmology

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* Equilibrium and Out of Equilibrium dynamics of highly excited strings and some cosmological implications

* String thermodynamics subject with a long history, yet

  many important open question

  of both formal and phenomenological character remain

* Let me start by listing two recent (of many) reviews:

  * Hagedorn String Thermodynamics in Curved Spacetimes and near Black Hole Horizons

    T. G. Mertens (PhD Thesis), 2015

  * Superstring Cosmology - A Complementary Review R Brandenberger, 2023

* Our focus: Boltzmann Equations, extracting Equilibration rates & connect to some cosmological observables dark radiation, high frequency gravitational waves

This talk is going to be based on

* A Frey, R Mahanta and AM

  (Phys.Rev.D 105 (2022) 6, 066007)

* A Frey, R Mahanta, AM, F Muia, F Quevedo, G Villa

  (JHEP 03 (2024). 112)

* A Frey, R Mahanta, AM, F Quevedo, G Villa

  (To Appear)

# Outline

- Boltzmann equation approach to string thermodynamics (in flat space)
- String in Thermal Equilibrium as a solution to the dark radiation problem
- A Stochastic Background of High Frequency Gravitational Waves from Thermal Open Strings

Third part mostly in Gonzalo Villa's Talk

Session B3, aula P100

# 1. Boltzmann Equations for Thermal Strings in flat space

* There are various approaches to study to thermal strings, the Boltzmann equation approach was pioneered in:
  D Lowe and L Thorlacius '94; S Lee and L Thorlacius '97
  E. J. Copeland, T. W. B. Kibble, and D. A. Steer '98

* Here, the basic idea is to write a rate equation for $n(\ell)$
  $n(\ell)d\ell$ : the number of strings of length $(\ell, \ell + d\ell)$
  ($\ell$ length of strings, defined as $\ell \equiv 2\pi\alpha' M$)

$$\boxed{\frac{\partial n(\ell, t)}{\partial t} = \text{Interaction Rates}; \ell >> 1}$$

* Note that are only keeping track of the number of strings at length $\ell$, so a coarse grained description.

We will touch upon three areas:

- The form of the interaction rates (& provide strong evidence that they admit a simple interpretation)
- The structure of detailed balance & equilibrium solutions
- And finally: Non-equilibrium Dynamics

with effectively Non-compact directions, with cosmological applications in mind.

J Manes '02

- Interaction rates
  - The rates can be determined by string perturbation theory. Here, we are working in the limit of $\ell >> 1$.
  - We have found strong evidence that they can be obtain by a <u>random walk picture of string interactions</u>.
- To exhibit this, I will focus on the decay of a highly excited closed string to two closed strings (other cases in the paper)

* The analytic form of the decay rate of a high excited closed string in a specific state $|S\rangle$ to two other such strings

$$|S\rangle \rightarrow |s'\rangle + |s''\rangle$$

is unknown.

* On the other hand, the decay rate for the inclusive process where one averages over all initial states of the same mass (length) and sums over final states with the same mass is available.
* These are exactly the kind of decay rates that one is interested if one wants to write a rate equation for $n(\ell)$.

* The decay rate per unit length of the outgoing string $\ell'$ is given by

$$\frac{d\Gamma}{d\ell'} \sim g_s^2 \ell \left( \frac{\ell}{\ell'(\ell - \ell')} \right)^{d/2}$$

in "d" non-compact directions.

* An important feature of these averaged processes is that the total string length is conserved

$$\ell = \ell' + \ell''$$

Next, let us give this decay rate a random walk interpretation

$$\frac{d\Gamma}{d\ell'} \sim g_s^2 \ell \left( \frac{\ell}{\ell'(\ell - \ell')} \right)^{d/2}$$

So, *What is the random walk interpretation ?* It is has four postulates

- String interactions rates are proportional to their length
- Long strings are in a random walk configuration.
- Interactions take place when strings intersect.
- Interaction rates are weighted by the probabilities for intersection.

Let us analyse our decay rate in this light

$$\frac{d\Gamma}{d\ell'} \sim g_s^2 \ell \left( \frac{\ell}{\ell'(\ell - \ell')} \right)^{d/2}$$

* There is the overall factor of $\ell$ as string interactions proportional to their length.

* What about the term in the brackets ?

  * For this, Recall: A random walk of length $\ell$ in “d" dimensions fills in a volume $\ell^{d/2}$

  * The probability that it closes on to itself is proportional to $\frac{1}{\ell^{d/2}}$.

* The probability that a string closes on to itself is proportional to $\frac{1}{\ell^{d/2}}$.

* With this, we can write the term is the brackets

$$\left(\frac{\ell}{\ell'(\ell-\ell')}\right)^{d/2} = \frac{\left(\frac{1}{\ell'}\right)^{d/2}\left(\frac{1}{\ell''}\right)^{d/2}}{\left(\frac{1}{\ell}\right)^{d/2}}$$

in terms of the closure probabilities of the three strings.

* Thus, the term in brackets is
<u>the probability that closed mother string self intersects such that there are daughter strings of length $\ell'$ and $\ell''$</u>

* So, the rate is in agreement with the random walk interpretation.

* This was one example, we have found that all interaction rates available in the literature are consistent with the random walk picture

* The next step, is to write the Boltzmann equation. In some situations, some of the interaction rates needed are not available from the string perturbation theory literature. Given the evidence for the random walk picture we use have used it to determine these interactions

* We plan verify these by explicit string perturbation theory computations. Making this connection more concrete is an interesting direction.

# Boltzmann Equations

**Open and Closed Strings in the presence of space-filling branes**

$$\frac{\partial n_c(l)}{\partial t} = +\frac{b}{2N} V_\perp \frac{n_o(l)}{l^{d/2}} - a\frac{N}{V_\perp} l n_c(l) + \frac{1}{2}\int_{l_c}^{l-l_c} dl' \left(\kappa_a \frac{n_c(l')l'\, n_c(l-l')(l-l')}{V} - \kappa_b l n_c(l) \left(\frac{l}{l'(l-l')}\right)^{d/2}\right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(\kappa_b l' n_c(l') \left(\frac{l'}{l(l'-l)}\right)^{d/2} - \kappa_a \frac{l n_c(l)(l'-l) n_c(l'-l)}{V}\right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(\kappa_c \frac{(l'-l)n_o(l')}{l^{d/2}} - \kappa_d \frac{l n_c(l)(l'-l)n_o(l'-l)}{V}\right).$$

$$\frac{\partial n_o(l)}{\partial t} = +a\frac{N}{V_\perp} l n_c(l) - \frac{b}{2N} V_\perp \frac{n_o(l)}{l^{d/2}} + \int_{l_c}^{l-l_c} dl' \left(\frac{b}{2NV_{||}} n_o(l') n_o(l-l') - a\frac{N}{V_\perp} n_o(l)\right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(2a\frac{N}{V_\perp} n_o(l') - \frac{b}{NV_{||}} n_o(l) n_o(l'-l)\right) + \int_{l_c}^{l-l_c} dl' \left(\kappa_d \frac{l'(l-l')n_c(l')n_o(l-l')}{V} - \kappa_c n_o(l) \frac{l-l'}{l'^{d/2}}\right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(\kappa_c \frac{n_o(l')l}{(l'-l)^{d/2}} - \kappa_d \frac{l n_o(l)(l'-l)n_c(l'-l)}{V}\right) + \text{(2-2 interactions)},$$

For equilibrium solutions set RHS to zero, we use detailed balance.

# Detailed Balance, Equilibrium Solutions Boltzmann Equations

* Detailed balance: Equilibrium Solutions to Boltzmann equations can be obtained by setting the net rate along every reaction channel to be zero.

* This usually yields that the number densities are Bose-Einstein or Fermi-Dirac

* For strings, this usual picture does not go through. As we keep on increasing the energy, string grow in size and can eventually fill the volume of the space multiple times

* Effectively, the phase space available to them changes.

* Evident from various well known results:
  Example: Total number of closed strings of length $(\ell, \ell + d\ell)$ when all directions are compact is

$$n(\ell)d\ell = \frac{e^{-\ell/L}}{\ell}d\ell$$

  Note: Number of strings is not extensive.

* We need to revisit detailed balance.

# The Structure of Detailed Balance

- We have found that detailed balance works "length by length" in the string interactions
- For the talk, our example: Closed strings with all direction compact directions

First equality from D Lowe and L Thorlacius '94

$$
\begin{aligned}
\frac{\partial n(\ell)}{\partial t} =& \frac{\kappa}{V}\left\{-\frac{1}{2}\ell^2 n(\ell) + \frac{1}{2}\int_0^\ell d\ell'\,\ell' n(\ell')(\ell-\ell')n(\ell-\ell')\right.\\
&\left.\quad -\ell n(\ell)\int_0^\infty d\ell'\,\ell' n(\ell') + \int_0^\infty d\ell'\,(\ell+\ell')n(\ell+\ell')\right\}\\
=& \frac{\kappa}{2V}\int_0^\ell d\ell'\ \left(\ell' n(\ell')(\ell-\ell')n(\ell-\ell') - \ell n(\ell)\right)\\
&\quad + \frac{\kappa}{V}\int_\ell^\infty d\ell'\ \left(\ell' n(\ell') - (\ell'-\ell)n(\ell'-\ell)\ell n(\ell)\right).
\end{aligned}
$$

Second line is our rewrite which makes the structure of detailed balance manifest

Organisation of interactions according to the length of the interacting string: with split into longer and shorter strings

$$\frac{\partial n(\ell)}{\partial t} = \frac{\kappa}{2V}\int_0^{\ell} d\ell' \ \left(\ell' n(\ell')(\ell-\ell')n(\ell-\ell') - \ell n(\ell)\right) + \frac{\kappa}{V}\int_{\ell}^{\infty} d\ell' \ \left(\ell' n(\ell') - (\ell'-\ell)n(\ell'-\ell)\ell n(\ell)\right).$$

- The first line: Interactions with strings shorter than $\ell$
  - fusion of strings $(\ell', \ell-\ell') \to \ell$ in the first term
  - self-decay in the second term, to all possible daughter strings
- The second line: interactions with strings longer than $\ell$
  - self decay of strings of length $\ell'$ into $\ell$ and $\ell'-\ell$
  - the inverse process of fusion of strings $(\ell, \ell'-\ell) \to \ell'$.

# Detailed Balance ...

Example: Closed strings with all direction compact directions

$$\begin{aligned}\frac{\partial n(\ell)}{\partial t} =& \frac{\kappa}{2V}\int_0^{\ell} d\ell' \, \left(\ell' n(\ell')(\ell-\ell')n(\ell-\ell') - \ell n(\ell)\right) \\ &+ \frac{\kappa}{V}\int_{\ell}^{\infty} d\ell' \, \left(\ell' n(\ell') - (\ell'-\ell)n(\ell'-\ell)\ell n(\ell)\right).\end{aligned}$$

This makes the channels manifest

* Channels are characterised by the value of $\ell'$
* Setting the integrand equal to zero gives us the equilibrium solutions.

The same philosophy works in more general settings: non-compact direction, D-branes ....

The same philosophy works in more general settings: non-compact direction, D-branes ....

* Closed strings in d “non-compact” directions:

$$n_c(\ell) = V \frac{\kappa_b}{\kappa_a} \frac{e^{-\ell/L}}{\ell^{1+d/2}} \,,$$

* Open and closed strings with space filling D branes

$$n_c(\ell) = V \frac{\kappa_b}{\kappa_a} \frac{e^{-\ell/L}}{\ell^{d/2+1}} \,, \qquad n_o(\ell) = \frac{2a\kappa_b N^2 V_\parallel}{b\kappa_a V_\perp} e^{-\ell/L} \,.$$

* Boltzmann Equations allow us to probe non-equilibrium dynamics Various cosmological applications ...

* I will discuss the case of closed strings in all compact directions in detail in the talk. Consider a perturbation $\delta n(\ell, t)$ about the equilibrium solution.

$$n(\ell, t) = \frac{e^{-\ell/L}}{\ell} + \delta n(\ell, t)$$

* This satisfies an integro-differential equation

$$\frac{V}{\kappa}\frac{\partial \delta n(\ell, t)}{\partial t} = -\left(\frac{\ell^2}{2} + \ell L\right)\delta n(\ell, t) + \int_0^l dl'\, \ell' \delta n(\ell', t)\left(e^{\frac{-(\ell-\ell')}{L}} - 1\right) - \delta E\left(e^{-\ell/L} - 1\right)\,,$$

where $\delta E$ is the energy of the perturbation,

$$\delta E \equiv \int_0^\infty d\ell'\, \ell' \delta n(\ell', t)\,.$$

The integro-differential equation:

$$\frac{V}{\kappa}\frac{\partial \delta n(\ell,t)}{\partial t} = -\left(\frac{\ell^2}{2} + \ell L\right)\delta n(\ell,t) + \int_0^l d\ell'\, \ell' \delta n(\ell',t)\left(e^{\frac{-(\ell-\ell')}{L}} - 1\right) - \delta E\left(e^{-\ell/L} - 1\right),$$

* Interestingly, it is possible to find explicit solutions. By taking derivatives, we show that $\delta n(\ell,t)$ needs to satisfy the differential equation

$$\left[2(\ell+L) + \left(\frac{\ell^2}{2} + \ell L + \frac{V}{\kappa}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial \ell}\right]\left(\delta n(\ell,t) + L\frac{\partial \delta n}{\partial \ell}\right) = 0\,.$$

* And thus the problem can be divided into two simpler problems:
  a) Find the kernel of the operator

$$\mathcal{L} \equiv 2(\ell+L) + \left(\frac{\ell^2}{2} + \ell L + \frac{V}{\kappa}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial l}\,,$$

  b) Translate the functions in the kernel, denoted $K(\ell,t)$, into fluctuations through a first order inhomogeneous ODE:

$$\delta n(\ell,t) + L\frac{\partial \delta n(\ell,t)}{\partial \ell} = K(\ell,t)\,,$$

# Non-Equilibrium Solutions

These have the form:

* $$\delta n(\ell, t) = e^{-\ell/L}$$

* $$\delta n_c(\ell, t) = \sqrt{\frac{\pi(c + tL^2)}{2}} \frac{e^{-\frac{\ell}{L} + A(t)^2}}{L} \mathrm{Erf}\left( A(t), A(t) + \sqrt{\frac{c + tL^2}{2}} \frac{\ell}{L} \right)$$

  where $\mathrm{Erf}(z_1, z_2) = \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-t^2} dt$ is the incomplete error function, and

  $$A(t) = \sqrt{\frac{c + tL^2}{2}} \left( 1 - \frac{1}{c + tL^2} \right) .$$

# Non-Equilibrium Solutions

* Only zero energy perturbations

$$\delta E \equiv \int_0^\infty d\ell' \, \ell' \delta n(\ell', t) \; = 0$$

  settle to the background equilibrium solutions.

* These can be obtained by considering linear combination of the two solutions

* They have a length dependent equilibration rates

$$\Gamma(\ell) = \frac{\kappa}{V} \left( \frac{\ell^2}{2} + \ell L \right)$$

  result in keeping with the basic estimates of D Lowe and L Thorlacius '94

* A similar approach can be used to study the dynamics in the presence of non-compact directions One finds

$$\Gamma(\ell) = \kappa \ell \frac{E}{V} + \dots$$

* For mixture of open and close string, the quantity $\delta n_c - \delta n_o$. has

$$\Gamma(\ell) \sim \ell \frac{N}{V_\perp}$$

where $\frac{N}{V_\perp}$ is the density of D-branes.

* Important inputs for cosmology

# 2. Thermal Strings and Dark Radiation

* $\Delta N_{\text{eff}}$: the energy density of new light (relativistic) degrees of freedom at the time of neutrino decoupling measured as an effective number of additional *thermal* neutrino-like species.

$$\Delta N_{\text{eff}} = \frac{\rho_{\rm dr}(t_\nu)}{\rho_{\nu-\text{add}}(t_\nu)}$$

* It is a powerful probe of additional species, independent of how they couple to matter. Bounds: $\Delta N_{\rm eff} < 0.3$.

* For theories where all constituents of the universe are produced from decay of a single species at early times (such as inflation or if the universe through an epoch of modulus domination)

$$\Delta N_{\text{eff}} = \frac{43}{7}\frac{\rho_{\rm dr}(t_{\rm rh})}{\rho_{\rm vis}(t_{\rm rh})}\left(\frac{g(T_\nu)}{g(T_{\rm rh})}\right)^{1/3} = \frac{43}{7}\frac{B_{\rm dr}}{B_{\rm vis}}\left(\frac{g(T_\nu)}{g(T_{\rm rh})}\right)^{1/3},$$

Formula from: Cicoli Conlon and Quevedo '12

* An understanding of the predictions for dark radiation in string compactifications is an important question.

## How Much Dark Radiation ?

* $$\Delta N_{\mathrm{eff}} = \frac{43}{7}\frac{B_{\mathrm{dr}}}{B_{\mathrm{vis}}}\left(\frac{g(T_{\nu})}{g(T_{\mathrm{rh}})}\right)^{1/3},$$

  Presents a tension. Order one branching ratios implies an overabundance of dark radiation.

* Model building primarily focussing on getting a low ratio for $\frac{B_{\mathrm{dr}}}{B_{\mathrm{vis}}}$.

# Hot Strings and $\Delta N_{\text{eff}}$

*
$$\Delta N_{\text{eff}} = \frac{43}{7}\frac{B_{\mathrm{dr}}}{B_{\mathrm{vis}}}\left(\frac{g(T_\nu)}{g(T_{\mathrm{rh}})}\right)^{1/3},$$

* The 1/3 power of the ratio of g-factors, does not lead to much suppression is the SM or GUT models.

* What about an epoch of "hot strings" (strings close to the Hagedorn temperature ) where there is a explosion in the number of degrees of freedom?

* Next, a concrete realisation of this idea
* The specific model will be in IIB, where high temperature string are produced at the end of inflation
* But, the key feature of the mechanism is entropic and thus can have general applicability.

One scenario for the universe entering an epoch of hot strings.

A Frey, R Myers and A Mazumdar '06

Setting:

- II B flux compactification
- The standard model is realised on D-branes located at the bottom of a warped throat.

A condition: The scale of inflation is much larger than the warped string scale (at the vacuum configuration of the throat) for the SM degrees of freedom

$$H_{\text{inf}} \gg e^{A} \frac{1}{\sqrt{\alpha'}}$$

If the above condition is met:

Phase of thermal open strings at end of inflation.

# Long Thermal Open Strings Phase with D3 Branes (in flat space)

S Lee and L Thorlacius '97 ; S Abel, J Barbon, I. Kogan and E. Rabinovici '99

- Consider strings at high temperature in a toroidal compactification with D3/anti-D3 branes

- The D-3 branes are space filling in Minkowski directions and point like in the compact directions.
- Analysing the thermodynamics of the system, one finds:

  - The energy density is dominated by open strings

  - The entropy of the open strings is given by:

$$S_{\mathrm{open}}(E) = \beta_H E + \sqrt{\frac{8 N_D^2 V_{\parallel} E}{m \mu^2 V_{\perp}}} \ ,$$

# Long Open Strings Phase with D3 Branes (in flat space)

$$S_{\text{open}}(E) = \beta_H E + \sqrt{\frac{8 N_D^2 V_\parallel E}{m\mu^2 V_\perp}} \, ,$$

where:

* $\beta_H = 2\pi\sqrt{2\alpha'}$ is the type IIB inverse Hagedorn temperature.
* $N_D$ the number of D-branes
* $\mu = 1/2\pi\alpha'$ is the string tension
* $V_\parallel$ and $V_\perp$ are the volumes parallel and perpendicular to the branes.
* $m$ is an order one constant

Above formula is flat space. Apply to branes at the bottom of a KS throat keeping in mind the following:

* The local string scale is warped string scale.

$$\sqrt{\alpha'} \to \ell_{\rm SM}$$

* Strings do not probe the entire compactification but only the region in the bottom of the throat. M Jackson, N Jones and J Polchinski '04

* The R-R flux tadpole can cancel due to contributions from other regions in the compactification.

* Define $g_E(T)$ by $\rho \equiv \pi^2 g_E(T) T^4/30$, for $T \to T_H$

$$g_E(T) \approx \frac{30 N_D^2}{\pi^4 m v} \frac{1}{(1 - T/T_H)^2} \, .$$

* The Entropy g-factor

$$g_S(T) = \frac{3}{4} g_E(T)$$

* The ratio $\frac{T_{rh}}{T_H}$ depends on the scale of inflation. For GUT scale inflation and $N_D \sim 10$, a conservative estimate yields

$$\left( \frac{g(T_\nu)}{g(T_{\rm rh})} \right)^{1/3} \lesssim 0.01$$

Recall $\Delta N_{\bf eff} = \frac{\bf 43}{\bf 7} \frac{\rm Br_{dr}}{\rm Br_{vis}} \left( \frac{g(T_\nu)}{g(T_{\rm rh})} \right)^{\bf 1/3}$ This can easily provide the necessary suppression for $\Delta N_{\rm eff}$

## 3. High Frequency Gravitation Waves from Thermal Open Strings

Gonzalo Villa

Session B3, aula P100

The setting

* The Standard Model realised by open string degrees of freedom
* In the very early universe, both closed and open strings were in thermal equilibrium at high temperatures

Graviton Production

* There are decay channels which lead to graviton production

$$\ell \to \ell' + g$$

* Q: What is the nature of the stochastic gravitational wave background that this produces ?

Q: What is the nature of the stochastic gravitational wave background that this produces ?

Will be described in in Gonzalo Villa's talk. Highlights:

- A characteristic spectrum with the frequency peak close to the CMB peak

- Amplitude: significantly higher than GW emission during a reheating epoch of the Standard model or BSM models !!

# Conclusions

* Boltzmann equations for highly excited strings: random walk picture, equilibration rates
* High Entropy as mechanism to suppress dark radiation.
* A characteristic spectrum for stochastic GW waves, which beats the SM

# Open directions

* A concrete relation between scattering of highly excited strings and the random walk picture.

* Moduli stabilisation, better understanding of string thermodynamics in curved backgrounds

* Under what circumstances does the universe enter into a phase of high temperature strings ?