Machine learning in CY geometry

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Based on collaborations with A. Lukas, F. Ruehle, R. Schneider (2111.01436, 2205.13408)
L. Anderson, J. Gray (2312.17125)
Y. Hendi, M. Walden (2406.----)

Why use ML in string theory?

- Build string vacuum with {Standard Model, dS, scale separation, ..}
  - Can ML pick good geometries? Speed up hard computations? Find vacua?
- Swampland program
  - Can ML help classify UV-complete effective field theories? Or test conjectures?
- Numerics: ML for conformal bootstrap, ML of CY metrics
- Learn mathematical structures (of relevance for physics)
- Physics-inspired models to explain how ML works

… *progress on all of these topics, driven by many researchers*

Reviews: Ruehle:20, Bao, He, Heyes, Hirst:22, Anderson, Gray, ML:23

- Talks by Stefano Lanza, Kit Fraser-Taliente, Justin Tan, Lucas Tsun Yin Leung,…

This talk in a nutshell

- ML is useful for CY geometry
  - Ricci flat metrics
  - Hermitian Yang-Mills connections
  - Harmonic representatives

See talks by Kit Fraser-Taliente and Justin Tan

- ML methods work for
  - generic CY manifolds
  - at given point in moduli space
- Symmetries can be encoded using invariant ML models

CY geometry: Ricci flat metrics

CY Theorem: Let X be an n-dimensional compact, complex, Kähler manifold with vanishing first Chern class.
Then in any Kähler class $[J]$, X admits a unique Ricci flat metric g_{CY}.

Calabi:54, Yau:78

- For n>1, *no analytical expression* for g_{CY}.

K3: Kachru-Tripathy-Zimet:18

- Solve $R_{ij}(g) = 0$ — 4^{th} order, non-linear PDE. Very hard.
- Equivalent to — 2^{nd} order PDE for function ϕ.
  Hard, but may solve numerically on examples

CY geometry: Ricci flat metrics

CY Theorem: Let X be an n-dimensional compact, complex, Kähler manifold with vanishing first Chern class.
Then in any Kähler class $[J]$, X admits a unique Ricci flat metric g_{CY}.

Kähler form J_{CY} satisfies

- $J_{CY} = J + \partial\bar{\partial}\phi$ — same Kähler class; ϕ is a function
- $J_{CY} \wedge J_{CY} \wedge J_{CY} = \kappa\, \Omega \wedge \bar{\Omega}$ — Monge-Ampere equation (κ constant) 2nd order PDE for ϕ
- Sample points on CY; compute J, Ω, κ; solve MA eq numerically

Numerical CY metrics

Algebraic CY metrics

- $K_k(z,\bar{z}) = \frac{1}{k}\sum \ln H_{a\bar{b}} p^a \bar{p}^{\bar{b}}$
  p^a spectral basis (polynomials)
- Solve for $H_{a\bar{b}}$ using
  - Donaldson algorithm
    Donaldson:05, Douglas-et.al:06, Douglas-et.al:08, Braun-et.al:08, Anderson-et.al:10, ...
  - Functional minimization
    Headrick–Nassar:13, Cui–Gray:20, Ashmore–Calmon–He–Ovrut:21
  - … or machine learning
    Anderson–et.al:20, Gerdes–Krippendorf:22,

Machine Learning CY metrics

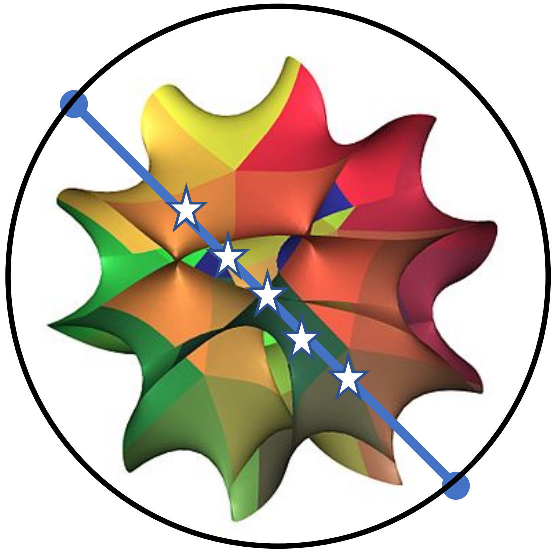
- Neural Networks are universal approximators
  Cybenko:89, Hornik:91, Leshno et.al:93, Pinkus:99,…
- Train ML model to approximate CY metric, or Kähler potential
  Ashmore–He–Ovrut:19, Douglas–Lakshminarasimhan–Qi:20, Anderson–et.al:20, Jejjala–Mayorga–Pena:20 , ML-Lukas-Ruehle-Schneider:21, 22 Ashmore–Calmon–He–Ovrut:21,22, Berglund-et.al:22 , Halverson-Ruehle:23, Douglas-Platt-Qi-24

1. Generate a point sample

On example CY need random set of points, sampled w.r.t. known measure

Leading algorithm: CY is hypersurface in $\mathbb{P}^n$ Douglas et. al: 06

- Sample 2 pts on $\mathbb{P}^n$, connect with line & intersect → $n+1$ pts
- Shiffman-Zelditch theorem: distributed w.r.t. $dvol_{FS}$



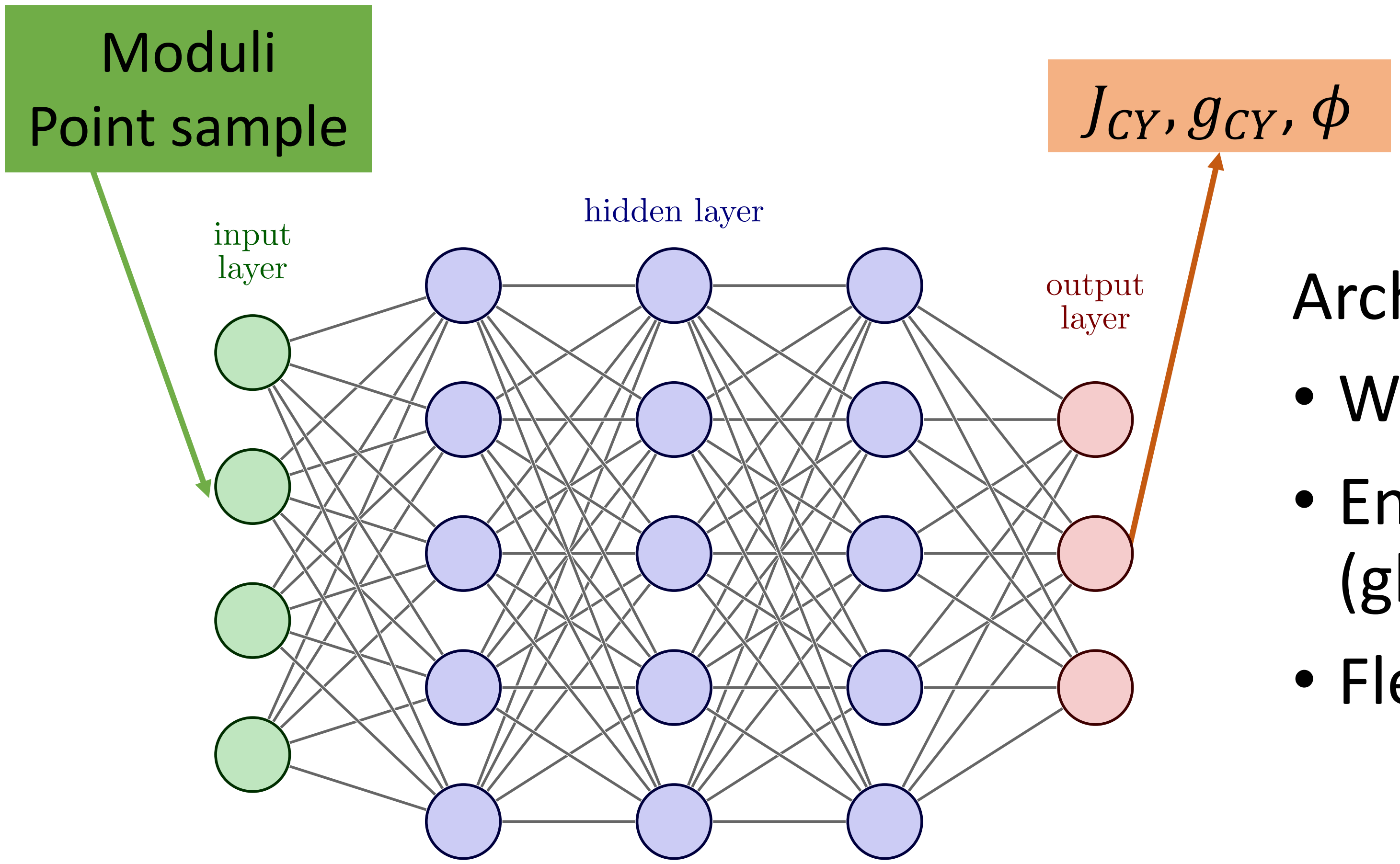
Generalizes to CICYs and CYs from Kreuzer-Skarke list

Douglas et.al: 07, ML, Lukas, Ruehle, Schneider: 21,22

- Fast point generators included in ML packages

MLgeometry, cymetric, cyjax

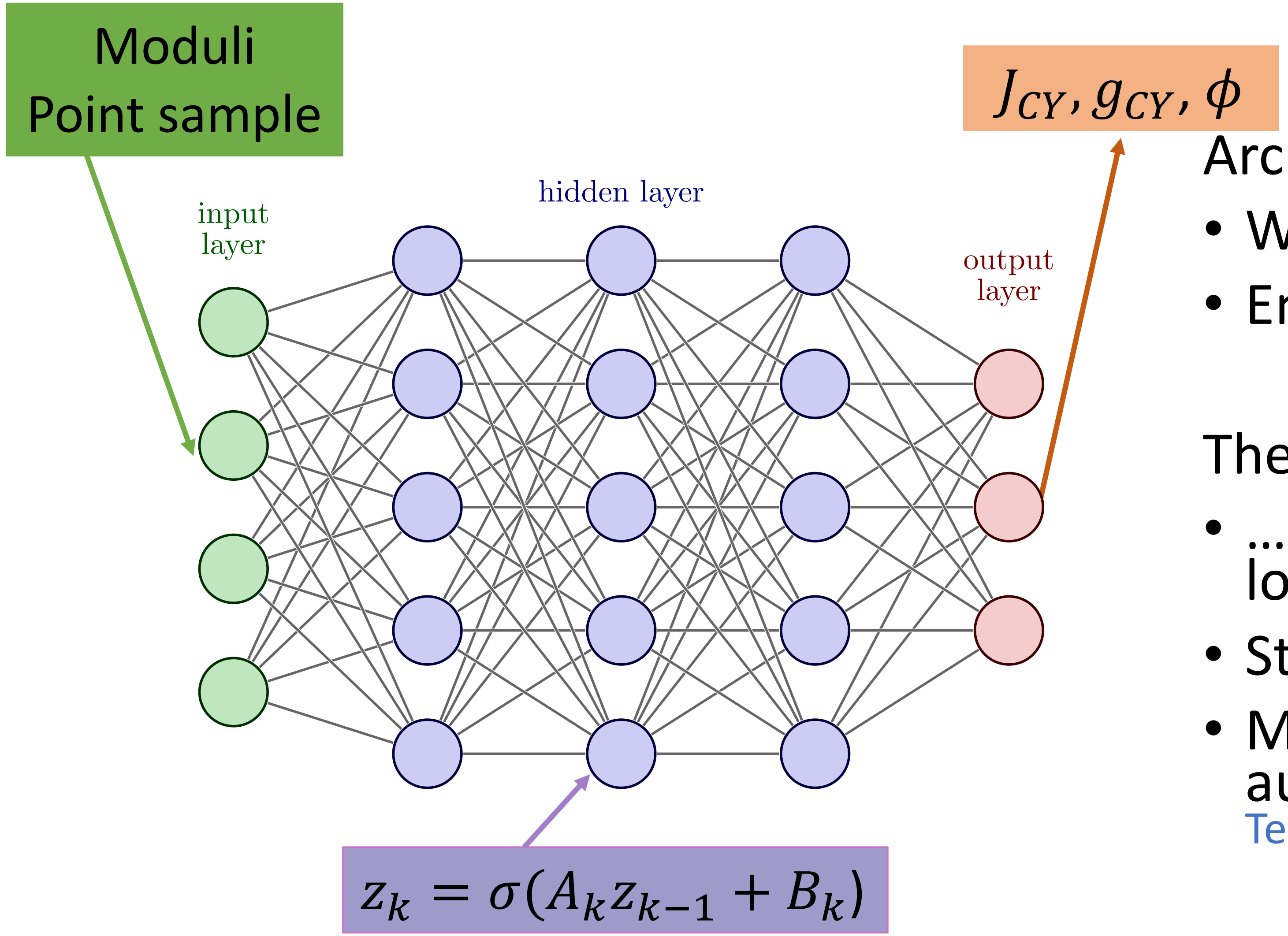
2. Set up the ML model



Architectural choices

- What to predict?
- Encode constraints in NN or loss? (global, complex, Kähler…)
- Flexibility vs. precision

3. Train the ML model



Architectural choices

- What to predict?
- Encode constraints in NN or loss?

Then train

- … i.e. adapt layer weights to minimize loss functions
- Stochastic gradient descent
- ML libraries w. highly optimized automatic differentiation
  TensorFlow, JAX, PyTorch

Loss functions encode math constraints

- Train the network to get unknown Ricci-flat metric (in given Kähler class)
- Use semi-supervised learning
  1. Encode mathematical constraints as custom loss functions
  2. Train network (adapt layer weights) to minimize loss functions
- Satisfy Monge-Ampere eq → minimize Monge-Ampere loss

$$\mathcal{L}_{\text{MA}} = \left\| 1 - \frac{1}{\kappa} \frac{\det g_{\text{pr}}}{\Omega \wedge \bar{\Omega}} \right\|_n$$

- Free metric ansatz → more loss functions (Kähler, transition, K-class)

4. Check accuracy

- After training, check that MA eq holds and Ricci tensor is zero

Check via established benchmarks:

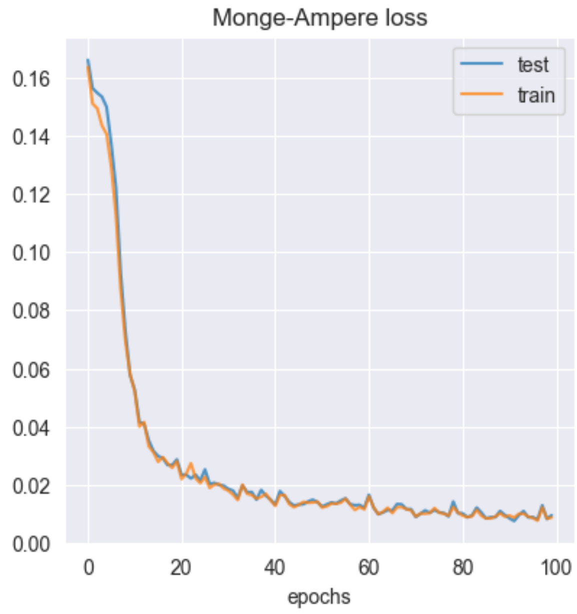
$$\sigma = \frac{1}{\mathrm{Vol}_{\mathsf{CY}}} \int_X \left| 1 - \kappa \, \frac{\Omega \wedge \overline{\Omega}}{(J_{\mathsf{pr}})^3} \right| \, , \; \mathcal{R} = \frac{1}{\mathrm{Vol}_{\mathsf{CY}}} \int_X |R_{\mathsf{pr}}| \, .$$

- Checking topological quantities, like volume and line bundle slopes, ensures metric prediction has good global properties .

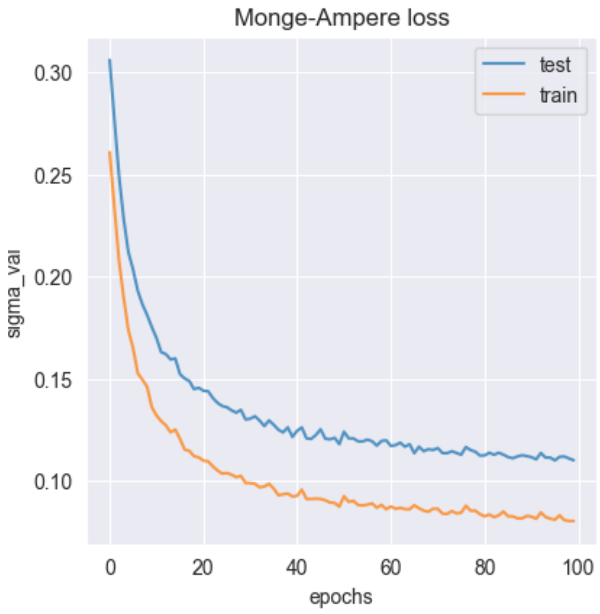
Experiments: Fermat vs. generic quintic

Anderson, Gray, ML:23

Monge-Ampere loss

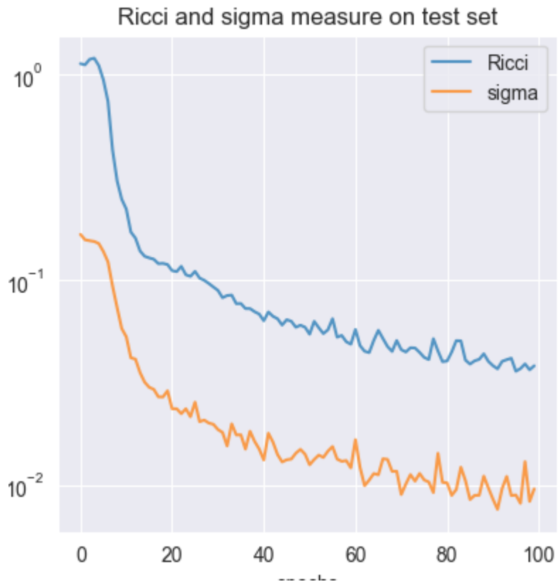


Fermat



Generic

Error measures



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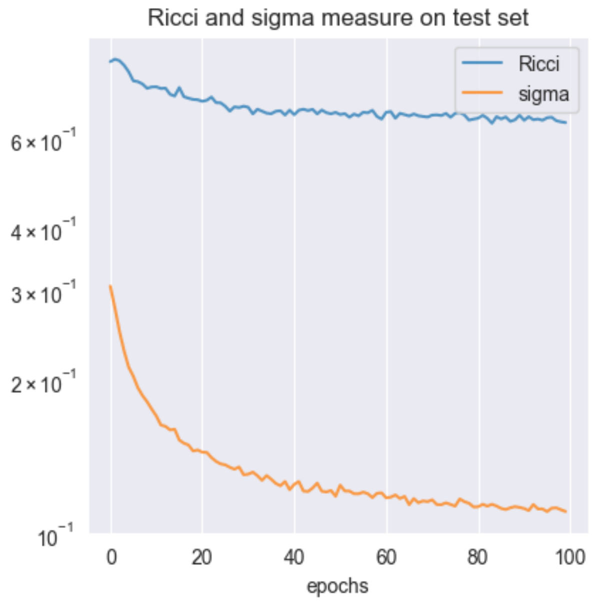


Generic

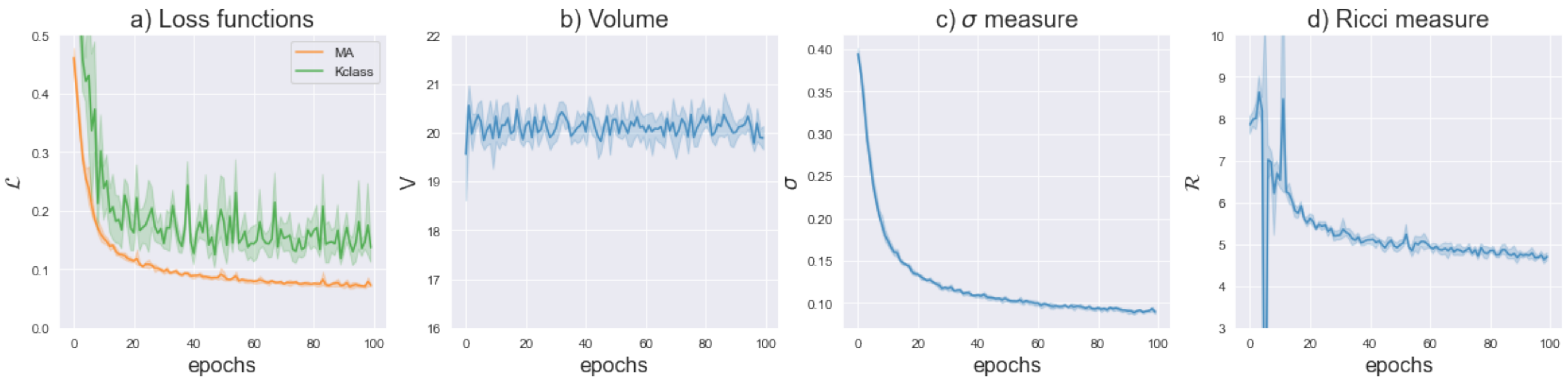
cymetric, ϕ-model, 100 000 points, 3 64-node layers, GELU, default loss parameters, Adam, batch (64, 50000)

ML methods are less sensitive to symmetry

Experiments: KS CY example

ML,Lukas, Ruehle,Schneider:22

- $h^{1,1} = 2, h^{2,1} = 80$ hypersurface from Kreuzer-Skarke database



cymetric, toric ϕ-model, default loss, 200 000 points

NN width 256, depth 3, GELU, batch (128, 10000), SGD w. momentum

ML methods work on both CICY and KS CYs

Accuracy and benchmarks

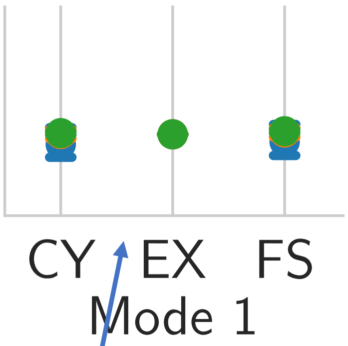
Ahmed & Ruehle:23

Accuracy improves with

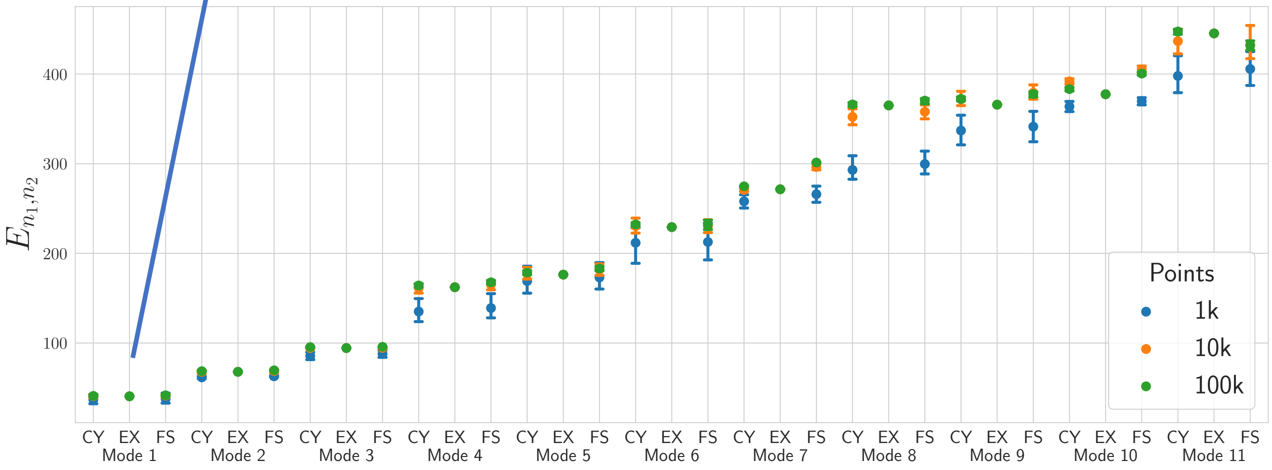
- Larger point sample
- Wider/deeper NN
- Train longer

and/or

- Change model architecture
  - cymetric ϕ-model + Spectral Layer
    Berglund et al:22, Butbaia et al:24



Benchmark cymetric
on the torus
Spectrum of Δ_{CY}



Application: Heterotic Standard-Like Models

Building blocks

- Ricci-flat Calabi Yau manifold X
- Vector bundle V satisfying Hermitian Yang-Mills eq.
- Discrete symmetry group G (break GUT)

- Many examples! E.g. 35 000 SLMs found with $V = \bigoplus L_i$
  Anderson et.al:11,12,13, …
  with RL/gen.alg. ML-Schneider:20, Constantin et.al: 21, Abel et al:21,23,…
- Next step toward SM: compute normalized Yukawa couplings
  Butbaia-et.al:24, Constantin-et.al:24

See talks by Kit Fraser-Taliente and Justin Tan

Application: Heterotic Standard-Like Models

Building blocks

- Ricci-flat Calabi Yau manifold X
- Vector bundle satisfying HYM eq.
- Discrete symmetry $G \rightsquigarrow$ smooth quotient CY X/G
  - allows to break GUT using Wilson lines
  - symmetries: permutations, discrete phase rotations, shifts of input z_i

- How can we restrict the model prediction to group invariant metrics?
- Can we use ML for Ricci flat metric on quotient CY?

ML G-invariant CY metrics

Hendi, ML, Walden:24 (work in progress)

- Let X be smooth CY, G symmetry, $g_{CY} = g_{FS} + \partial\bar{\partial}\phi$
- ML model which approximates $\phi(z)$ is G-invariant if

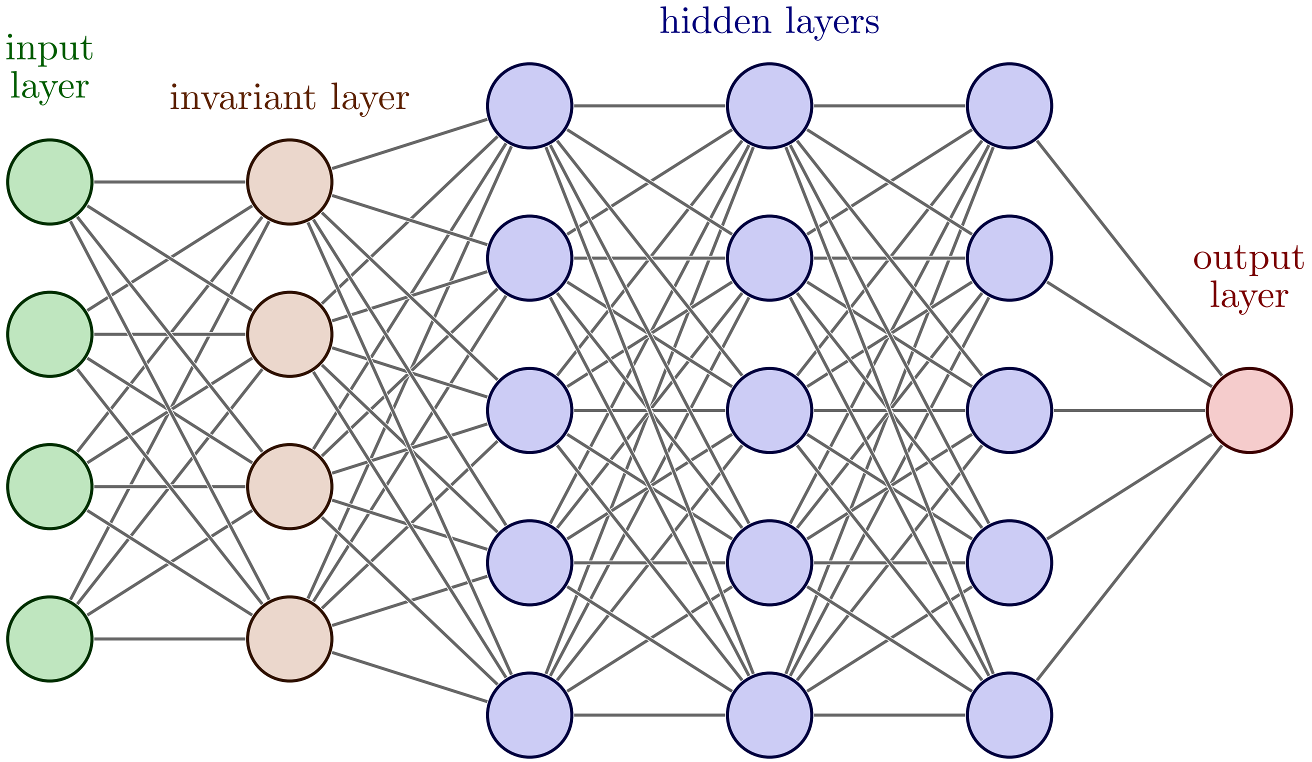
$$\phi(g \cdot z) = \phi(z)$$

- With spectral basis, ϕ invariant if expanded in invariant polynomials
  Donaldson:05, Headrick-Nassar:13, Douglas et al:08, …
- For ML, any G-invariant layer makes model invariant
  - Invariant NNs are Universal approximators for invariant functions Yarotsky:22,..
  - Invariant ML models can be constructed in many ways
  - Geometric Deep Learning: symmetry, performance & interpretability
    Bronstein et al:17,21,..

Invariance through non-trainable layers

Hendi, ML, Walden:24 (work in progress)

- Expand input in group invariant polynomials
  Yarotsky:22
  $\rightsquigarrow$ Spectral layer
- G-canonicalization: Invariant layer projects input to fundamental domain of G
  Aslan, Platt, Sheard:22, Kaba et.al. 23

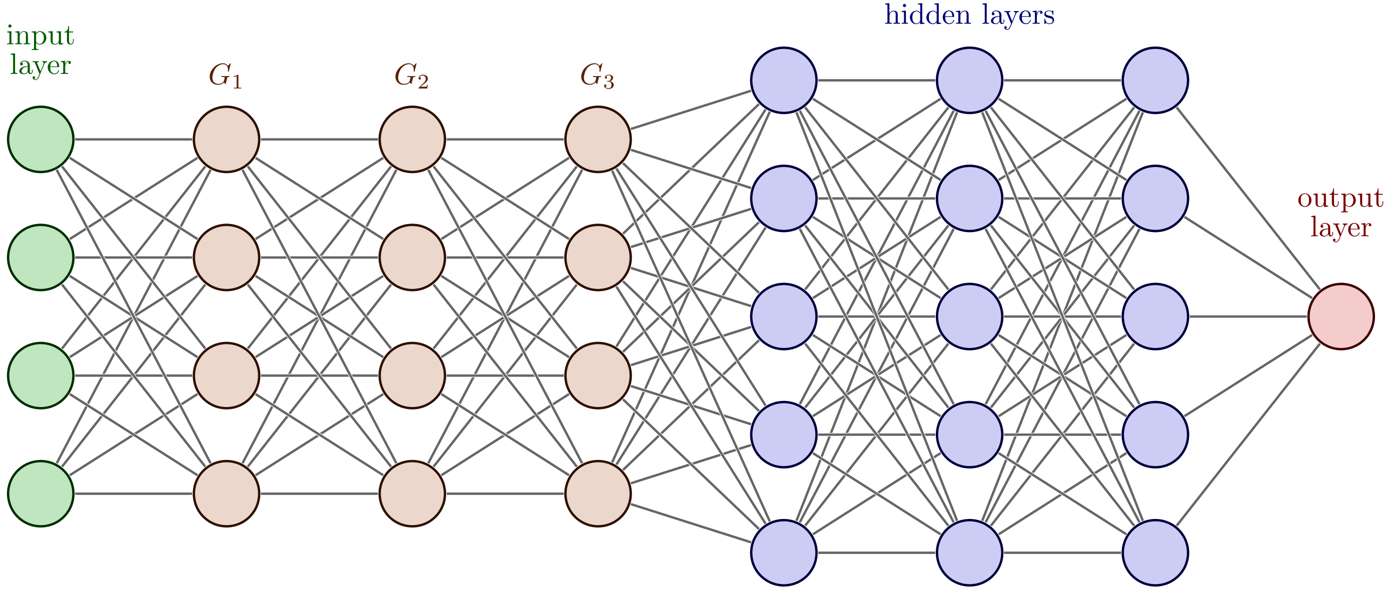


Invariance through non-trainable layers

Hendi, ML, Walden:24 (work in progress)

G-canonicalization:

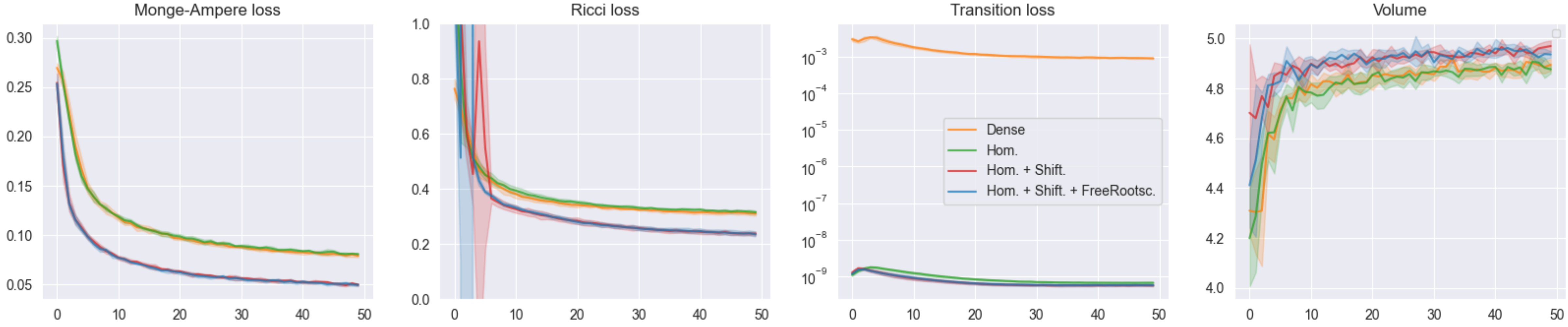
- Invariant layer projects data to fundamental domain of G
- Modular and stackable (given compatibility condition)
- Easily included in ML models for CY metrics (we use cymetric)



CY metric on smooth quintic quotient

Hendi, ML, Walden: 24 (work in progress)

- Ricci-flat metric on $X/\mathbb{Z}_2^2$
- Symmetries (shifts and phase rotation) act freely
- ϕ-model of cymetric with 2 canonicalization layers



Conclusion and outlook

- Can learn Ricci flat metrics on CICY and KS CY manifolds, for given moduli.
- Mathematical constraints: encoded in NN or in loss functions
- Performant ML packages: cymetric, MLgeometry, cyjax,…
- Architecture determines accuracy, performance, generality.

Applications and generalizations

  - Yukawa couplings Butbaia-et.al:24, Constantin-et.al:24
  - Swampland distance conjecture Ashmore:20, Ashmore & Ruehle:21 Ahmed & Ruehle:23
  - Moduli-dependent CY metrics Anderson-et.al:20, Gerdes-Krippendorf:22
  - Warped CY metrics, G-structure geometry
  - (towards) G2 metrics Douglas-Platt-Qi:24

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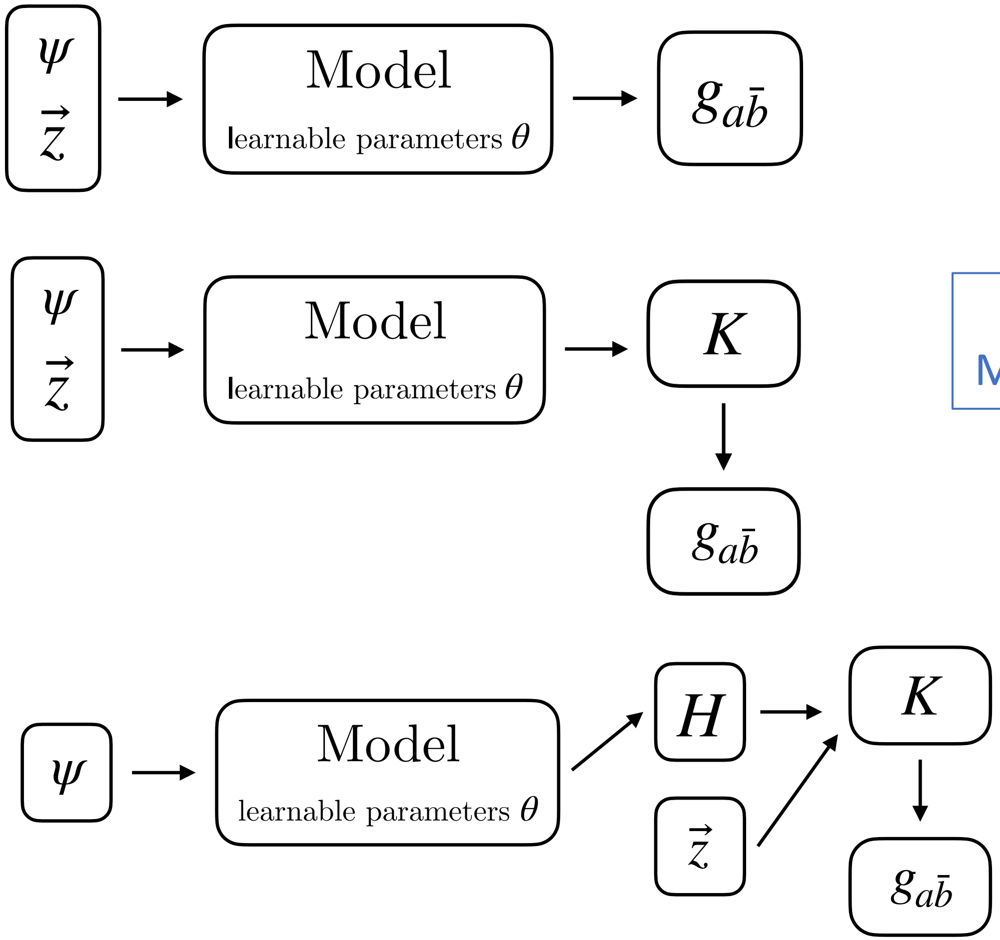
Thank you for listening!

Additional slides

ML models - choice of architecture

1. Learn metric
   Anderson-et.al.:20,
   Jejjala–Mayorga–Pena:20
   ML-Lukas-Ruehle-Schneider:21, 22
2. Learn Kähler potential (ϕ)
   Anderson-et.al.:20,
   Douglas–Lakshminarasimhan–Qi:20,
   Ashmore–Calmon–He–Ovrut:21,22,
   ML-Lukas-Ruehle-Schneider:21, 22,
   Berglund-et.al.:22
3. Learn Donaldson's H matrix
   Anderson-et.al.:20,
   Gerdes–Krippendorf:22



cymetric

cymetric
MLgeometry

cyjax

Figure adapted from Anderson et al:20

Homogeneous rescaling invariance

- Often work with homogeneous coordinates of ambient space
- Rescaling invariant $\phi(z) = \phi(\lambda\, z)$
- True *exactly* for algebraic metric, using spectral basis
  Anderson et al : 20, Douglas et al : 20, Gerdes & Krippendorf:22, ...
- The models of the cymetric package are only approximately invariant.
- Combining cymetric ϕ −model with "spectral layer" gives invariant model
  Berglund et al:22

$$(z_0, \ldots, z_n) \mapsto \begin{pmatrix} \frac{z_0\,\bar{z}_0}{|z|^2} & \frac{z_0\,\bar{z}_1}{|z|^2} & \cdots & \frac{z_0\,\bar{z}_n}{|z|^2} \\ \frac{z_1\,\bar{z}_0}{|z|^2} & \frac{z_1\,\bar{z}_1}{|z|^2} & \cdots & \frac{z_1\,\bar{z}_n}{|z|^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{z_n\,\bar{z}_0}{|z|^2} & \frac{z_n\,\bar{z}_1}{|z|^2} & \cdots & \frac{z_n\,\bar{z}_n}{|z|^2} \end{pmatrix}$$

Application: Heterotic Standard-Like Models

Building blocks

- Ricci-flat Calabi Yau manifold X
- Vector bundle V satisfying Hermitian Yang-Mills eq.

$$F \wedge \Omega = 0 = F \wedge J \wedge J$$

- Discrete symmetry (break GUT)

- Hard differential equation. Requires g_{CY}
- Solution exists iff V is polystable
  Donaldson:85, Uhlenbeck–Yau:86
- Can solve numerically…
  Douglas et.al.:06 Anderson et.al:10,11
- … with ML
  Ashmore–Deen–He–Ovrut:21
  ML-Lukas-Ruehle-Schneider:22,
  Butbaia et.al:24, Constantin et. al:24

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- Next step: compute normalized Yukawa couplings

See talks by Cristoforo Fraser-Taliente and Justin Tan