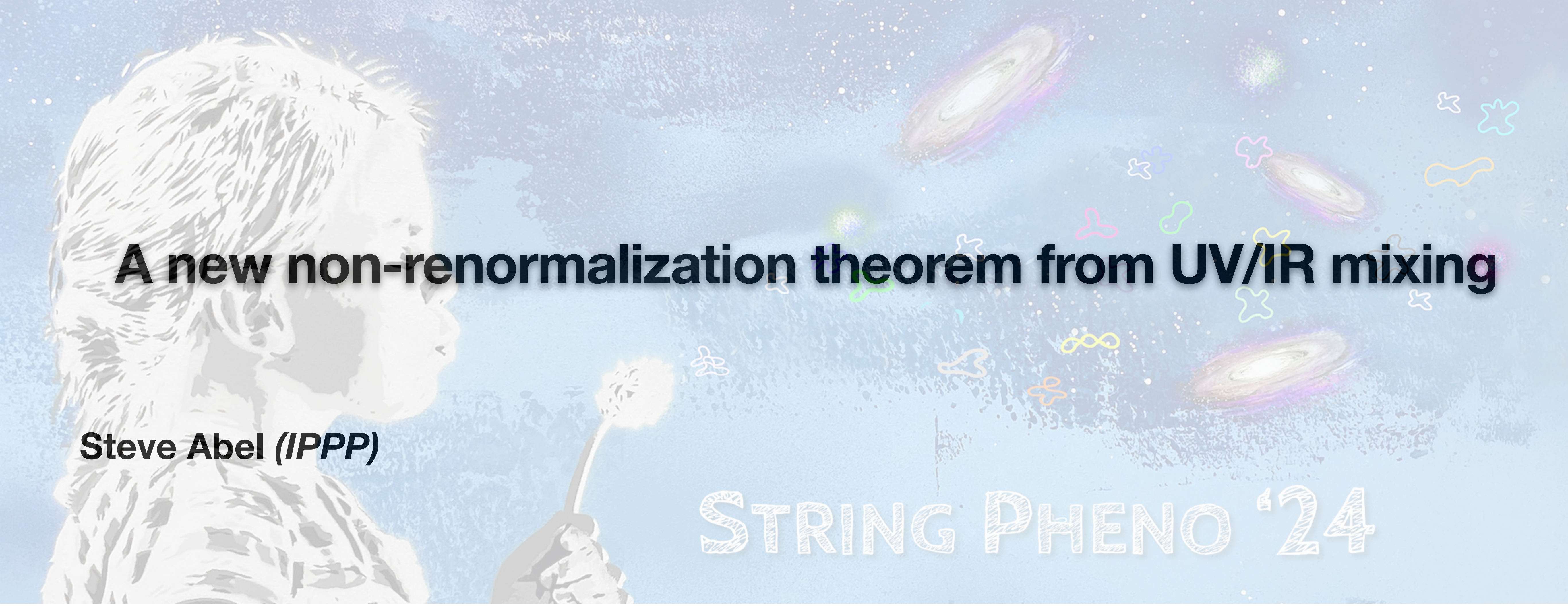
A new non-renormalization theorem from UV/IR mixing

Steve Abel *(IPPP)*



Based on the following set of papers on NON-SUPERSYMMETRIC strings …

- **w/ Dienes and Nutricati — arXiv:2406.NNNNN**
- w/ Dienes and Nutricati — *Phys.Rev.D* 107 (2023) 12, 126019 ; arXiv:2303.08534
- w/ Keith Dienes — *Phys.Rev.D* 104 (2021) 12, 126032; arXiv:2106.04622
- w/ Dienes+Mavroudi *Phys.Rev.*D 97 (2018) 12, 126017, arXiv: 1712.06894
- w/ Stewart, *Phys.Rev.D* 96 (2017) 10, 106013 arXiv:1701.06629
- Aaronson, SAA, Mavroudi, *Phys.Rev.D* 95, (2016) 106001, arXiv:1612.05742
- SAA JHEP 1611 (2016) 085, arXiv:1609.01311
- w/ Dienes+Mavroudi *Phys.Rev. D91*, (2015) 126014, arXiv:1502.03087

Themes of this talk …

There is a whole raft of supertrace identities associated with large distance limits, a.k.a. dimensions that can decompactify, that has not been noticed before.

These are true for any closed string theory (or theory with modular invariance)

These identities seem to have profound implications: they forbid power law running! (Non-renormalisation theorem)

More generally an aspect of UV/IR mixing

Outline

- An old but remarkable super trace identity
- Higher dimensions
- Theories with higher dimensional limits
- No power-law running

An old but remarkable supertrace identity

Only assumption of this talk: suppose as in closed string that finiteness is ensured by Modular Invariance …

The theory is defined by the modular invariant *string partition function* $Z(\tau)$ where modular invariant: requires $Z(\tau) = Z(\tau')$ where

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}$$

In principle $Z(\tau)$ holds all the information about the spectrum: also (theorem) in a 4D theory it can always be written

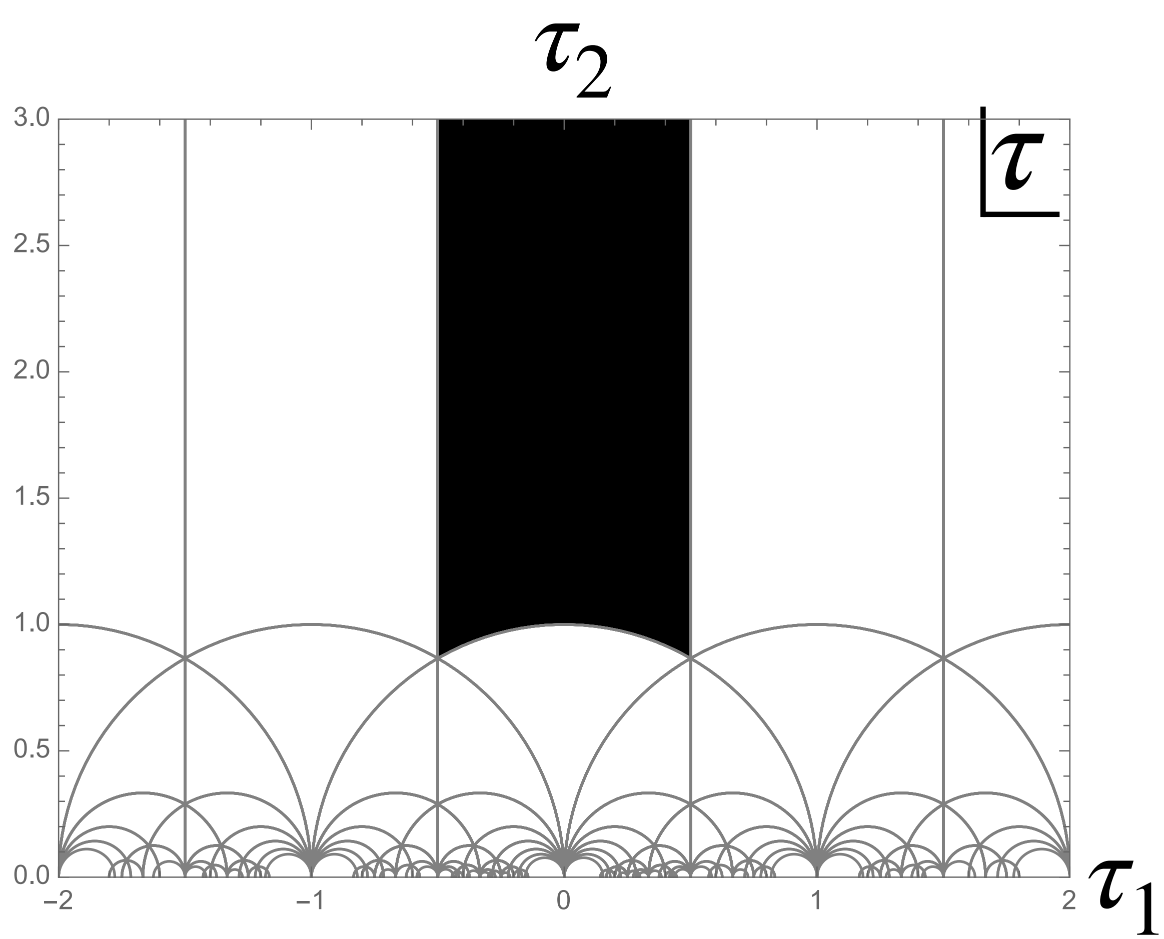
$$Z(\tau) = \frac{1}{\tau_2} \sum_{nm} a_{nm} \, q^n \bar{q}^m$$

where $q = e^{2\pi i \tau}$ and $\tau = \tau_1 + i\tau_2$. Think of this as the Fourier series of $Z(\tau)$.

This symmetry mixes UV/IR maximally:
To see this in action let's consider vacuum energy in any such theory:

$$\overset{(4)}{\Lambda} = -\frac{\mathcal{M}^4}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, Z(\tau)$$

This integral is *finite* because (as long as there are no physical tachyons) $Z(\tau)$ dies at large τ_2 and $\mathscr{F}$ does not contain the $\tau_2 \to 0$ point

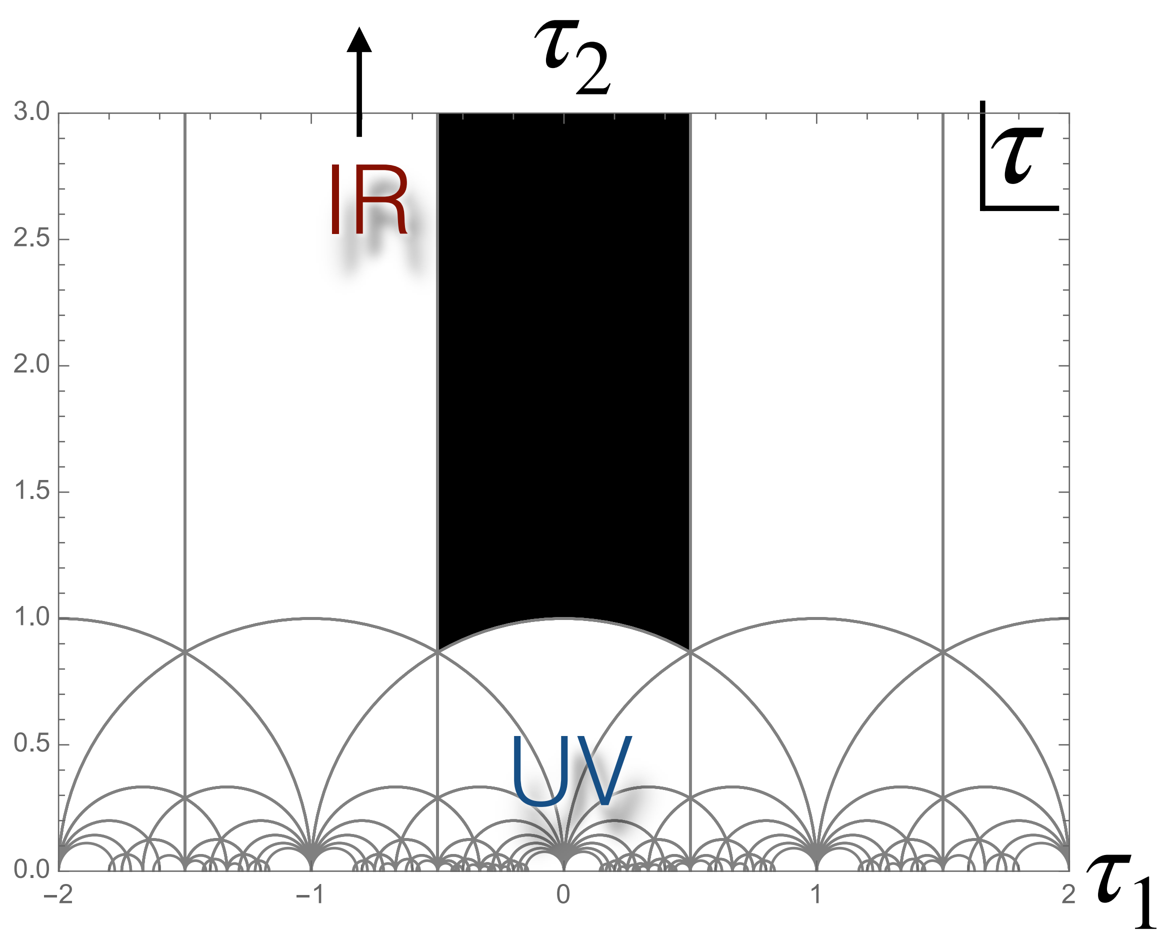


A bit of notation: $\left(\mathcal{M}^2 = \frac{1}{4\pi^2\alpha'} = \frac{M_s^2}{4\pi^2}\right)$

This symmetry mixes UV/IR maximally:
To see this in action let's consider vacuum energy in any such theory:

$$\Lambda^{(4)} = -\frac{\mathcal{M}^4}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3}\sum_{n,m} a_{nm}\, q^m \bar{q}^n$$

This integral is *finite* because (as long as there are no physical tachyons) $Z(\tau)$ dies at large τ_2 and $\mathscr{F}$ does not contain the $\tau_2 \to 0$ point

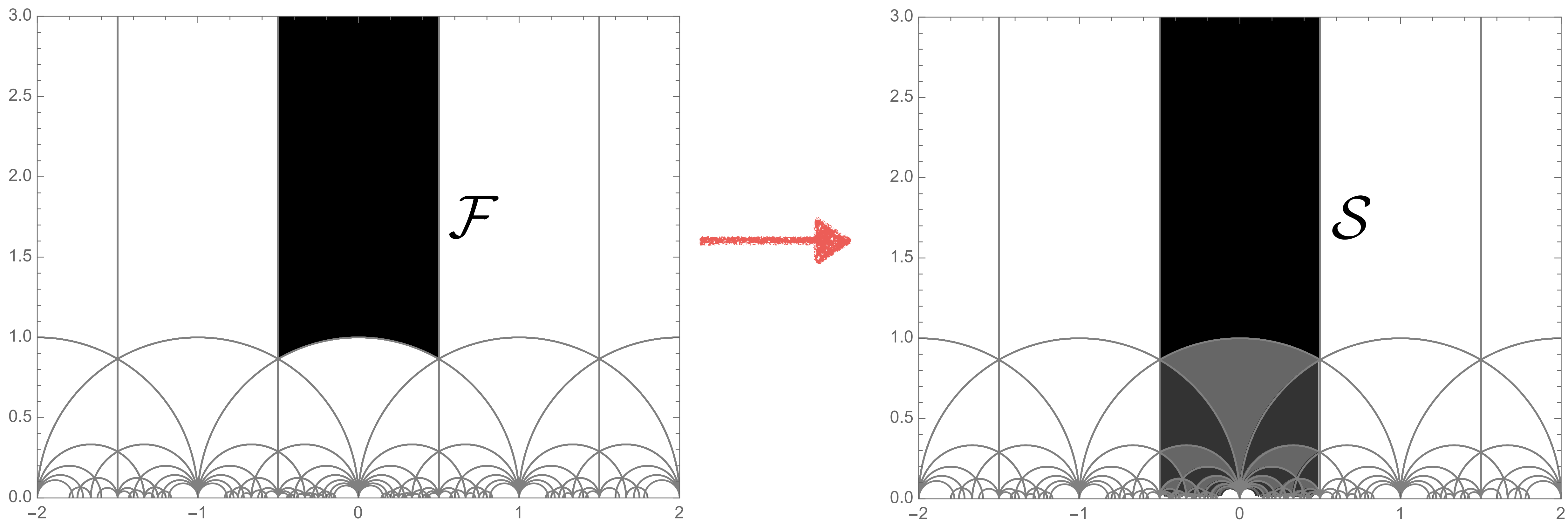


A bit of notation: $\left(\mathcal{M}^2 = \frac{1}{4\pi^2\alpha'} = \frac{M_s^2}{4\pi^2}\right)$

However: a method due to Rankin and Selberg (1939/40) lets us write this finite integral in another way — namely in terms the nett density of **physical (level-matched) states** — given by (Fourier zero-mode) integral:

$$
\begin{aligned}
g(\tau_2) &= -\frac{\mathcal{M}^4}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \; Z(\tau) \\
&= -\frac{\mathcal{M}^4}{2}\tau_2^{-1} \sum_{\text{states}} (-1)^F e^{-\pi\tau_2\alpha' M^2_{\text{state}}}
\end{aligned}
$$

RS use a transform to unfold $\mathscr{F}$ to the critical strip $\mathscr{S}$



Then using RS we see that ultimately the integral can be written as a limit …

$$-\frac{\mathcal{M}^4}{2}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}Z(\tau,\overline{\tau}) = \frac{\pi}{3}\,\underset{s=1}{\mathrm{Res}}\int_0^\infty d\tau_2\tau_2^{s-2}g(\tau_2)$$

Inverse Mellin transform

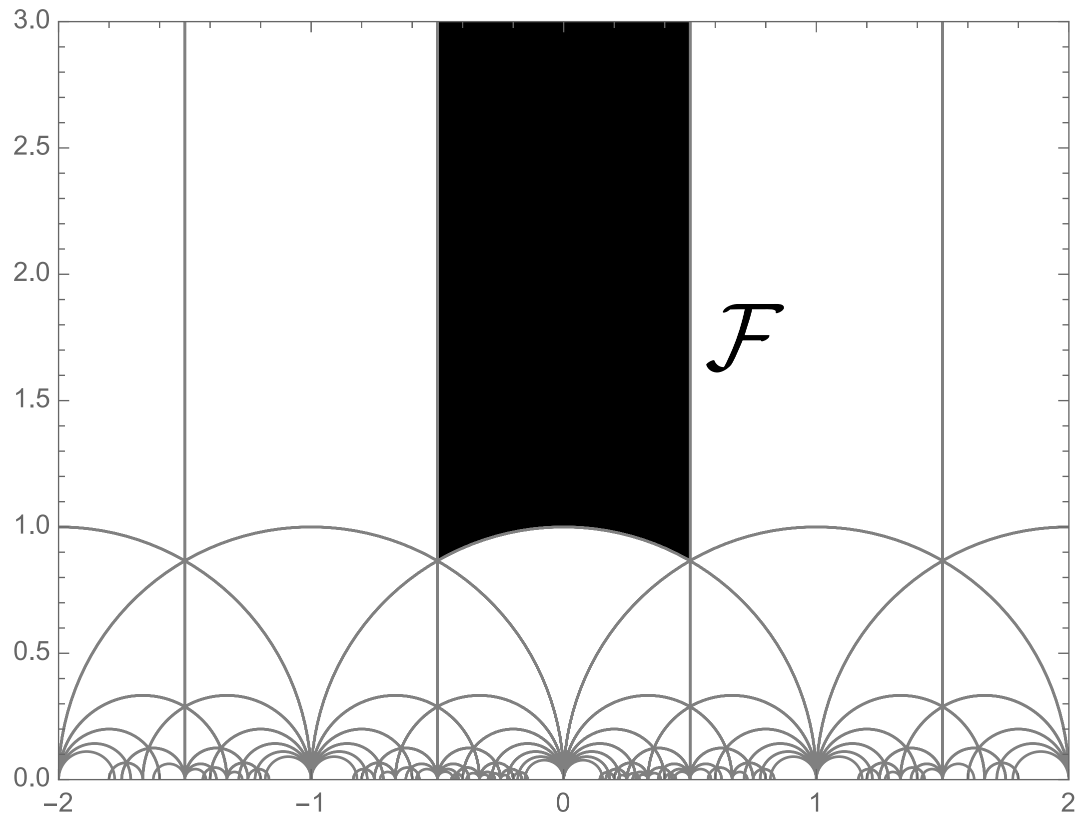
$$= \frac{\pi}{3}\lim_{\tau_2\to 0} g(\tau_2)$$

- Rankin, Selberg
- Zagier (1981)

In string theory: Kutasov, Seiberg; McClain, Roth, O'Brien, Tan; Dienes; Angelantonj, Florakis, Pioline, Rabinovici

Let's pause for a minute to see (as physicists) why this is remarkable:

There is a clear mapping from $\pi\alpha'\tau_2$ to the Schwinger parameter t when $\tau_2 \geq 1$: by naively integrating over the fundamental domain, we physicists see a result mimicking the EFT …

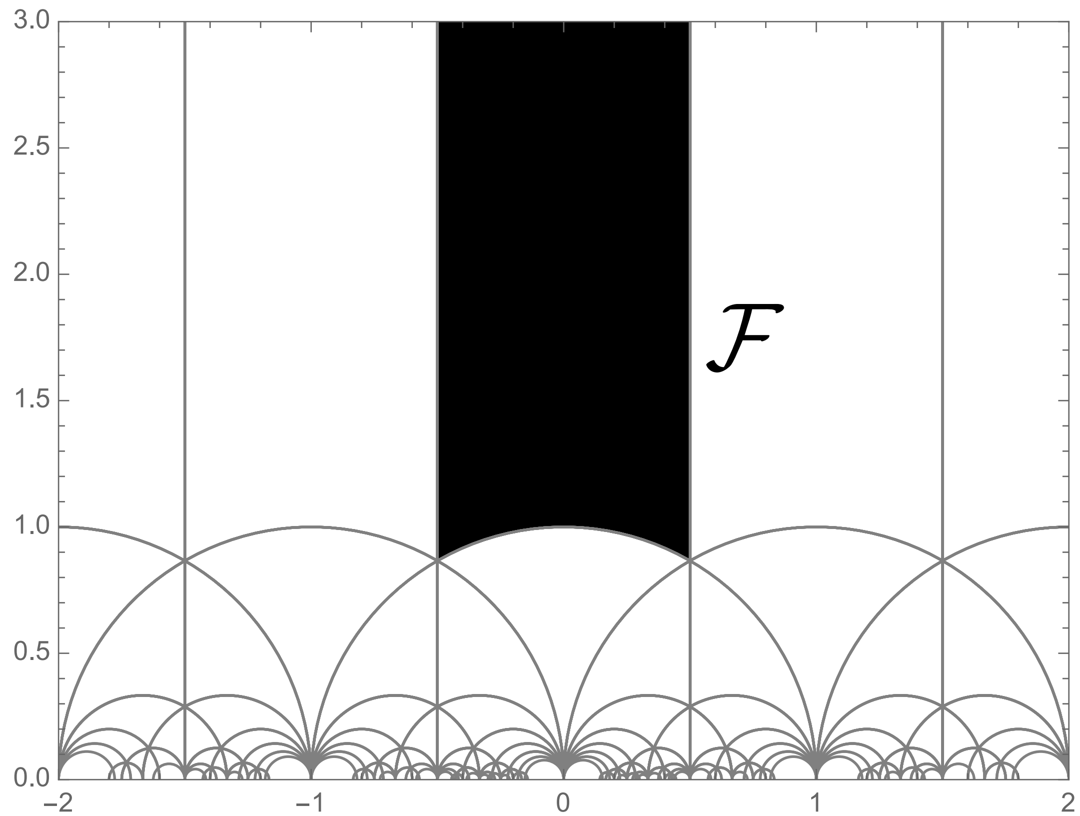


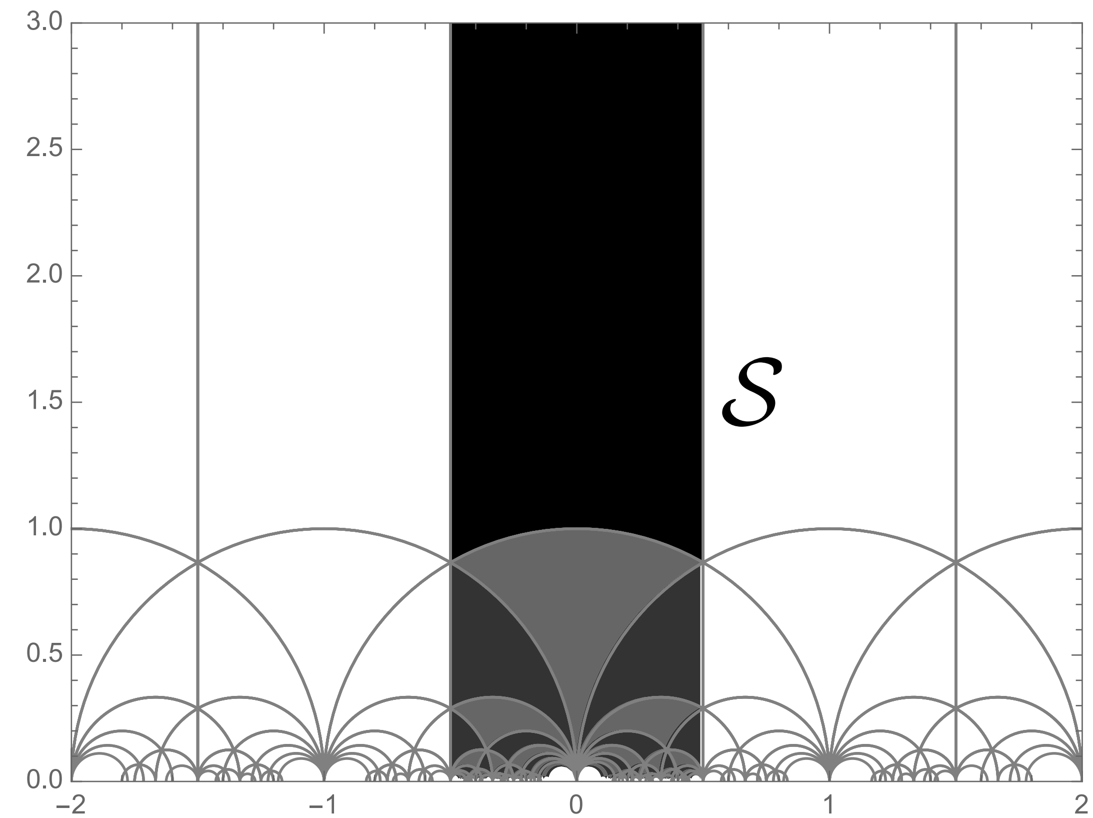
$$\Lambda^{(4)} \approx \int_1^\infty \frac{d\tau_2}{\tau_2^2} g(\tau_2)$$

$$\approx -\frac{\mathcal{M}^4}{2} \int_1^\infty \frac{d\tau_2}{\tau_2^3} \sum_{\text{states}} (-1)^F e^{-\pi\tau_2\alpha' M_{\text{state}}^2}$$

Let's pause for a minute to see (as physicists) why this is remarkable:

But according to RS this result is equal to a ***very not EFT-like limit*** - if anything it looks like a deep UV limit!!





$$\Lambda^{(4)} \approx \int_1^\infty \frac{d\tau_2}{\tau_2^2} g(\tau_2)$$

$$= \frac{\pi}{3} \lim_{\tau_2 \to 0} g(\tau_2)$$

$$\approx -\frac{\mathcal{M}^4}{2} \int_1^\infty \frac{d\tau_2}{\tau_2^3} \sum_{\text{states}} (-1)^F e^{-\pi \tau_2 \alpha' M_{\text{state}}^2}$$

This is the ultimate UV/IR mixing. But there is more …

Let's try and evaluate this RS limit:

$$\frac{\pi}{3}\lim_{\tau_2\to 0} g(\tau_2) \;\; = -\frac{\mathcal{M}^4}{2}\lim_{\tau_2\to 0}\sum_{\text{states}}(-1)^F\frac{1}{\tau_2}e^{-\pi\tau_2\alpha' M^2_{\text{state}}}$$

It looks like it diverges because of the $1/\tau_2$ prefactor in $Z(\tau_2)$!!!

This is the ultimate UV/IR mixing. But there is more …

Let's try and evaluate this RS limit:

$$\frac{\pi}{3} \lim_{\tau_2 \to 0} g(\tau_2) \;\; = -\frac{\mathcal{M}^4}{2} \lim_{\tau_2 \to 0} \sum_{\text{states}} (-1)^F \frac{1}{\tau_2} e^{-\pi \tau_2 \alpha' M^2_{\text{state}}}$$

It looks like it diverges because of the $1/\tau_2$ prefactor in $Z(\tau_2)$!!! … *Unless* …

$$\lim_{\tau_2 \to 0} \sum_{\text{states}} (-1)^F e^{-\pi \tau_2 \alpha' M^2_{\text{states}}} = 0$$

Thus — if we define a ***regulated supertrace*** appropriate for infinite towers of states for any operator $\mathcal{X}$,

$$\mathrm{Str}\,\mathcal{X} = \lim_{\tau_2\to 0} \sum_{\text{states}} (-1)^F \mathcal{X}_{\text{state}}\, e^{-\pi\tau_2\alpha' M^2_{\text{state}}} = 0$$

then here (where $\mathcal{X} = const$) *we see that any* modular invariant 4D theory with a finite Λ obeys a super trace relation

$$\boxed{\mathrm{Str}\,\mathbf{1} \;=\; 0}$$

Any tachyon-free modular invariant theory in 4D has $\mathrm{Str}(\mathbf{1}) = 0$ ***even when no SUSY!***

- Dienes, Misaligned SUSY, 1994

Or to put it another way … if we expand the density of states $g(\tau_2)$ around $\tau_2 = 0$ it generically would go like

$$g(\tau_2) = \frac{1}{\tau_2} \times \left(C_0 + C_1\tau_2 + C_2\tau_2^2 + \ldots\right)$$

but a modular invariant theory must have $C_0 = 0$ and must instead go like

$$g(\tau_2) = \frac{1}{\tau_2} \times \left(C_1\tau_2 + C_2\tau_2^2 + \ldots\right)$$

(For the sake of argument I will assume an integer expansion but only the first non-zero term needs to be an integer)

Thus we can get the answer, i.e. $\Lambda = \pi C_1/3$, by expanding the exponential around τ_2 and picking off the first term C_1:

$$\Lambda^{(4)} = \frac{1}{24}\mathcal{M}^2 \mathrm{STr} M^2$$

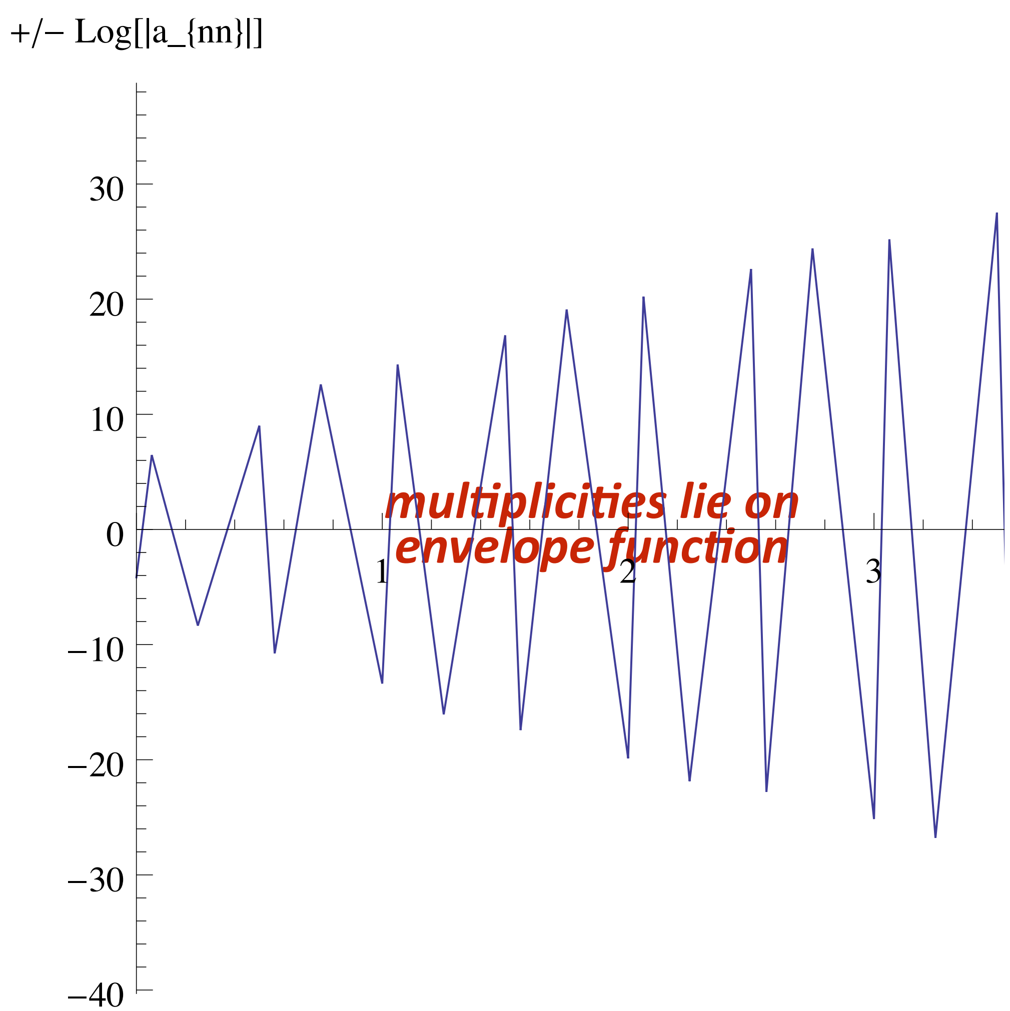
- Dienes, Misaligned SUSY, 1994
- Kutasov, Seiberg, 1994
- Dienes, Moshe, Myers 1995

Thus we can get the answer, i.e. $\Lambda = \pi C_1/3$, by expanding the exponential around τ_2 and picking off the first term C_1:

$$\Lambda^{(4)} = \frac{1}{24}\mathcal{M}^2 \mathrm{STr} M^2$$

- Dienes, Misaligned SUSY, 1994
- Kutasov, Seiberg, 1994
- Dienes, Moshe, Myers 1995

This looks like the quadratically divergent term in field theory but this definitely is *not* a field theory object — this supertrace is over the *infinite* string tower of states!!



- This crazy spectrum has finite Λ !!!

Higher dimensions

In theories with $D > 4$ space-time dimensions things get more constrained. The reason why is that the partition function takes the form

$$Z^{(D)}(\tau) \;=\; \tau_2^k \sum_{m,n} a_{mn}\, \overline{q}^m q^n$$

where $k = 1 - D/2$.

(We can see it has to go like this to give the Schwinger integral the right powers of t)

But now applying Rankin-Selberg we see that in a theory with $D = 4 + \delta$ …

$$g(\tau_2) = \frac{1}{\tau_2^{1+\delta/2}} \times \left(C_0' + C_1'\tau_2 + C_2'\tau_2^2 + \ldots\right)$$

$\Longrightarrow$ we have $C_0', C_1', \ldots, C_{\delta/2}' = 0$

Thus in a theory with $D = 4 + \delta$ expanding the expression for $\Lambda^{(D)}$ we have

$$\mathrm{Str}' M^k = 0$$

for all $k < 2 + \delta$ and

$$\Lambda^{(4+\delta)} = (-1)^{\delta/2} \frac{\pi}{6(1+\delta/2)!} \mathcal{M}^2 \mathrm{Str}' M^{2+\delta}$$

In $D = 4$ only the cosmological constant leads to a constraint, namely $\mathrm{Str}(\mathbf{1}) = 0$. But in higher dimensions there are less trivial amplitudes that get constrained: let's now extend the discussion to more general $\langle \mathcal{X} \rangle$; …

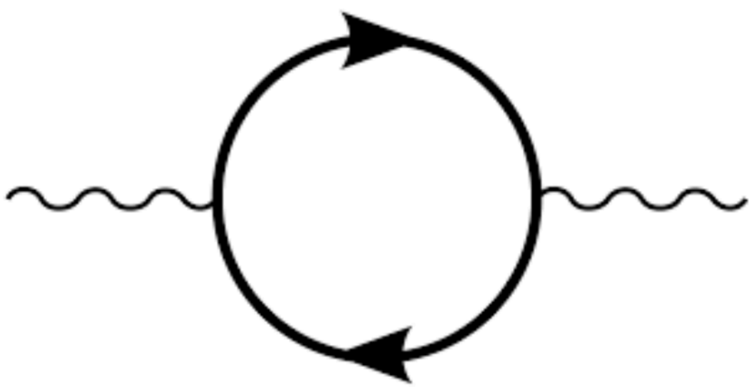
$$Z_{\mathcal{X}}^{(D)}(\tau) = \tau_2^k \sum_{n,m} a_{nm} \mathcal{X}_{nm} \bar{q}^m q^n$$

The operator $\mathcal{X}$ can be determined by modular completing the field theory operator.

In $D = 4$ only the cosmological constant leads to a constraint, namely $\mathrm{Str}(\mathbf{1}) = 0$. But in higher dimensions there are less trivial amplitudes that get constrained: let's now extend the discussion to more general $\langle \mathcal{X} \rangle$ …

$$Z_{\mathcal{X}}^{(D)}(\tau) = \tau_2^k \sum_{n,m} a_{nm} \mathcal{X}_{nm} \bar{q}^m q^n$$

The operator $\mathcal{X}$ can be determined by modular completing the field theory operator. Thus for example vacuum polarisation …



$$\mathcal{X} = \mathbb{X}_0 + \tau_2 \mathbb{X}_1 + \tau_2^2 \mathbb{X}_2$$

$$\begin{cases} \mathbb{X}_0 = 0 \\ \mathbb{X}_1 \equiv \frac{\xi}{2\pi}\left(\overline{Q}_H^2 - \frac{\overline{E}_2}{12}\right) \\ \mathbb{X}_2 \equiv -2\left(\overline{Q}_H^2 - \frac{\overline{E}_2}{12}\right) Q_G^2 \end{cases}$$

Space-time helicity Gauge charges

For example in a 6 dimensional theory we find a *constraint* plus a one - loop contribution to $16\pi^2/g_G^2 = 16\pi^2/g_{\text{tree}}^2 + \Delta_G$:

$$\mathrm{Str}'\,\overline{Q}_H^2 - \tfrac{1}{12}\mathrm{Str}'_E\,\mathbf{1} \;=\; 0$$

and …

$$\Delta_G \;\approx\; \frac{\pi}{3}\times\left[-2\,\mathrm{Str}'\,(Q_G^2\overline{Q}_H^2) + \frac{1}{6}\mathrm{Str}'_E\,Q_G^2 - \frac{\xi}{2\pi}\mathrm{Str}'\left(\overline{Q}_H^2\widetilde{M}^2\right) + \frac{\xi}{24\pi}\mathrm{Str}'_E\,\widetilde{M}^2\right]$$

where $\widetilde{M}^2 \;\equiv\; \dfrac{M^2}{4\pi\mathcal{M}^2}$

Theories with higher dimensional limits

***So the question is* — what happens when a 4 dimensional theory has a decompactification limit to a higher dimensional theory?**

***So the question is* — what happens when a 4 dimensional theory has a decompactification limit to a higher dimensional theory?**

Consider a toy factorisable example:

$$Z_{\mathcal{X}}^{(4)} = Z_{\mathcal{X}}^{(\text{base})} \cdot Z_{\text{KK/winding}}$$

where the "base" theory is the states without their KK and winding modes

$$Z_{\mathcal{X}}^{(\text{base})} = \tau_2^{-1} \sum_{mn}{}' a_{mn}\, \mathcal{X}_{mn}\, \overline{q}^m q^n$$

and $Z_{\text{KK/winding}}$ are all the KK and winding mode factors of $q, \bar{q}$.
e.g. for $\delta = 1$ it might be a simple circle factor of the form:

$$Z_{\text{KK/winding}} = \sum_{\tilde{m},\tilde{n}\in\mathbb{Z}} \overline{q}^{(\tilde{m}/\tilde{R}-\tilde{n}\tilde{R})^2/4}\, q^{(\tilde{m}\tilde{R}+\tilde{n}/\tilde{R})^2/4}$$

We can rearrange the contributions into modular invariant factors …

$$Z_{\mathcal{X}}^{(4)} = \left(\tau_2^{-\delta/2} \, Z_{\mathcal{X}}^{(\text{base})}\right) \cdot \left(\tau_2^{\delta/2} \, Z_{\text{KK/winding}}\right)$$

By examining the product at large radius the KK/winding factor turns into a simple volume factor …

$$Z_{\text{KK/winding}} \approx \frac{\mathcal{M}^{\delta} V_{\delta}}{\tau_2^{\delta/2}}$$

We can rearrange the contributions into modular invariant factors …

$$Z_{\mathcal{X}}^{(4)} = \left(\tau_2^{-\delta/2} \, Z_{\mathcal{X}}^{(\text{base})}\right) \cdot \left(\tau_2^{\delta/2} \, Z_{\text{KK/winding}}\right)$$

By examining the product at large radius the KK/winding factor turns into a simple volume factor …

$$Z_{\text{KK/winding}} \approx \frac{\mathcal{M}^{\delta} V_{\delta}}{\tau_2^{\delta/2}}$$

Putting the pieces together, at large volume we see that the 4D theory tends to the $4+\delta$ dimensional theory times a volume factor:

$$\lim_{V_\delta \to \infty} \left[\frac{1}{\mathcal{M}^{\delta} V_{\delta}} \, Z_{\mathcal{X}}^{(4)}\right] = \tau_2^{-\delta/2} \, Z_{\mathcal{X}}^{(\text{base})} = Z_{\mathcal{X}}^{(4+\delta)}$$

But at this point we notice a clash! … we know $Z_{\mathcal{X}}^{(4+\delta)}$ has to satisfy many more *constraints* than the four dimensional theory

The only way to resolve this clash and for *physics to be smooth* at infinite radius is for all the constraints to *already* be satisfied in the 4D theory … it turns out this is independent of the compactification radius:

The 4D theory will inherit the precise stricter internal cancellations of any higher-dimensional theory to which can be decompactified.

But at this point we notice a clash! … we know $Z_{\mathcal{X}}^{(4+\delta)}$ has to satisfy many more *constraints* than the four dimensional theory

The only way to resolve this clash and for *physics to be smooth* at infinite radius is for all the constraints to *already* be satisfied in the 4D theory … it turns out this is independent of the compactification radius:

> *The 4D theory will inherit the precise stricter internal cancellations of any higher-dimensional theory to which can be decompactified.*

For example $16\pi^2 g_G^{-2} = 16\pi^2 g_{\text{tree}}^{-2} + \Delta_G$ in a theory with $\delta = 2$ decompactification:

$$\begin{aligned} \text{Str}'\,\overline{Q}_H^2 - \tfrac{1}{12}\text{Str}'_E\,\mathbf{1} \;&=\; 0 \\ \Delta_G \;&\approx\; \frac{\pi}{3}\widetilde{V}_T\left[-2\,\text{Str}'\,(Q_G^2\overline{Q}_H^2) + \frac{1}{6}\text{Str}'_E\,Q_G^2 - \frac{\xi}{2\pi}\text{Str}'\left(\overline{Q}_H^2\widetilde{M}^2\right) + \frac{\xi}{24\pi}\text{Str}'_E\,\widetilde{M}^2\right] \end{aligned}$$

Generally we can expect a theory that can decompactify to look like this:

$$Z^{(4)} = \sum_{i=1}^{N} Z_i' \Theta_i$$

The i indicates a sum over different sectors … each with a “base” contribution Z_i' multiplying KK/winding factors Θ_i which turn into volumes at large radius.

Generally we can expect a theory that can decompactify to look like this:

$$Z^{(4)} = \sum_{i=1}^{N} Z'_i \Theta_i$$

The i indicates a sum over different sectors … each with a "base" contribution Z'_i multiplying KK/winding factors Θ_i which turn into volumes at large radius.

When δ dimensions become large in some direction labelled α, some the Θ_i factors contribute to a modular invariant sum $\Theta^{(\alpha)} = \sum c_i^{(\alpha)} \Theta_i$ yielding what we call the ***T*-volume** with the remaining contributions going exponentially fast to zero:

$$\widetilde{V}_T^{(\alpha)} \equiv \frac{3}{\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Theta^{(\alpha)} \sim \mathcal{M}^\delta V_\delta^{(\alpha)}$$

Generally we can expect a theory that can decompactify to look like this:

$$Z^{(4)} = \sum_{i=1}^{N} Z_i' \Theta_i$$

The i indicates a sum over different sectors … each with a "base" contribution Z_i' multiplying KK/winding factors Θ_i which turn into volumes at large radius.

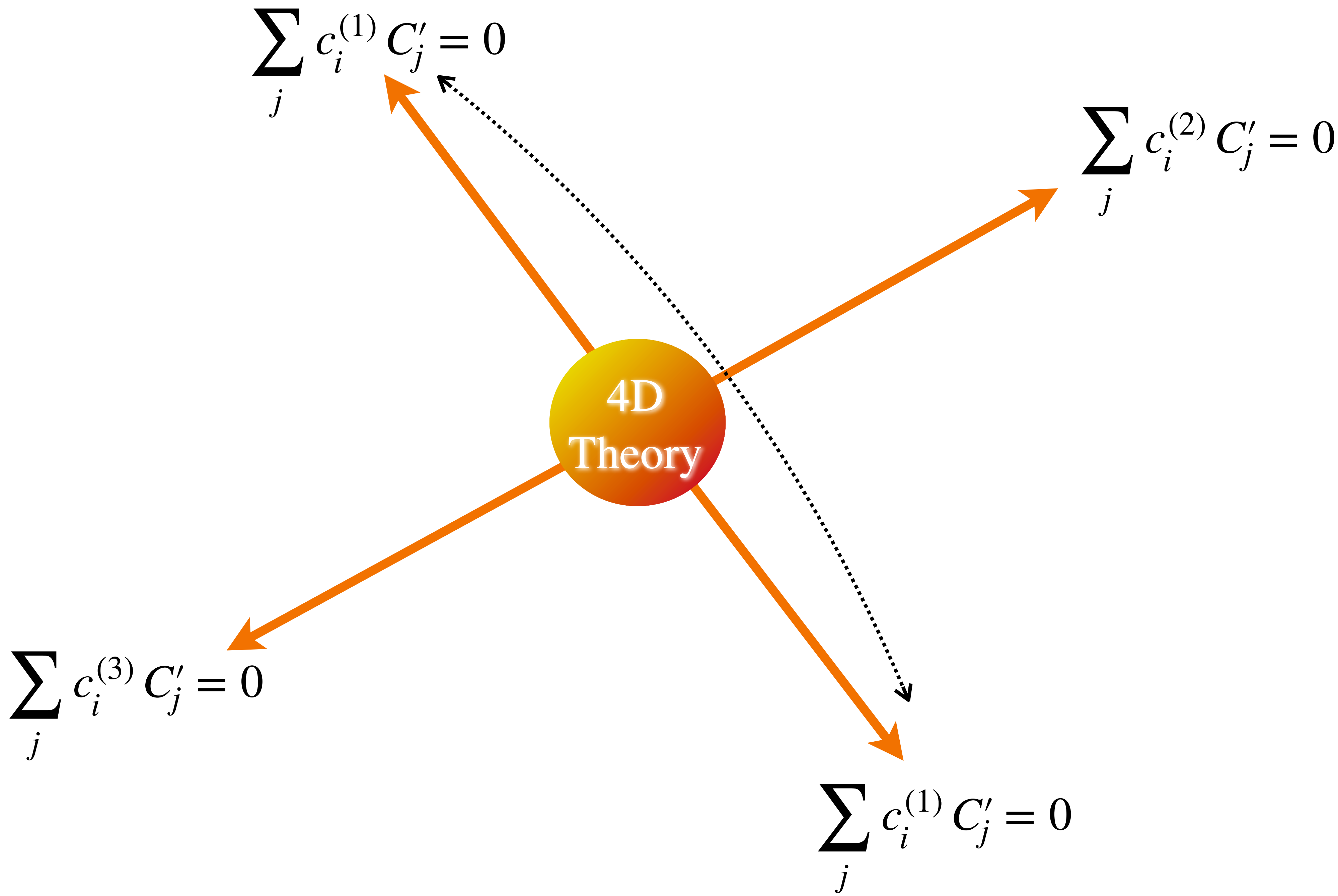
When δ dimensions become large in some direction labelled α, some the Θ_i factors contribute to a modular invariant sum $\Theta^{(\alpha)} = \sum c_i^{(\alpha)} \Theta_i$ yielding what we call the ***T*-volume** with the remaining contributions going exponentially fast to zero:

$$\widetilde{V}_T^{(\alpha)} \equiv \frac{3}{\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Theta^{(\alpha)} \sim \mathcal{M}^\delta V_\delta^{(\alpha)}$$

↑

RS derived modular invariant volume

So the cartoon looks like this …



Some of these endpoint theories related by T-duality transformations - but they all lead to a constraint that has to be satisfied in the 4D theory.

Wait : isn't this
just the usual $\mathcal{N} = 2$
SUSY in sectors from 2D
toroidal factors thing?

Wait : isn't this just the usual $\mathcal{N}=2$ SUSY in sectors from 2D toroidal factors thing?

Er … who mentioned SUSY? This is true in any modular invariant theory!

Wait : isn't this just the usual $\mathcal{N} = 2$ SUSY in sectors from 2D toroidal factors thing?

Er … who mentioned SUSY? This is true in any modular invariant theory!

These constraints are on supertraces so in pheno they can be approximate right?

Wait : isn’t this just the usual $\mathcal{N}=2$ SUSY in sectors from 2D toroidal factors thing?

Er … who mentioned SUSY? This is true in any modular invariant theory!

These constraints are on supertraces so in pheno they can be approximate right?

No, if the theory is UV finite it must be modular invariant — they must remain exact

Wait : isn't this just the usual $\mathcal{N}=2$ SUSY in sectors from 2D toroidal factors thing?

Er … who mentioned SUSY? This is true in any modular invariant theory!

These constraints are on supertraces so in pheno they can be approximate right?

No, if the theory is UV finite it must be modular invariant — they must remain exact

But why do we care?

Wait : isn't this just the usual $\mathcal{N} = 2$ SUSY in sectors from 2D toroidal factors thing?

Er … who mentioned SUSY? This is true in any modular invariant theory!

These constraints are on supertraces so in pheno they can be approximate right?

No, if the theory is UV finite it must be modular invariant — they must remain exact

But why do we care?

Let me show you …

No power-law running

Why can we now expect a statement on power-law running?

It helps to think of this from an equivalent extra-dimensional field theory view point - To see how RG running *emerges* we can insert an energy scale μ by putting a cut-off function $\mathscr{G}(\mu, t)$ into the one-loop Schwinger integral (c.f. Polchinski/Wetterich Exact RG):

$$\langle \mathcal{X} \rangle_{\rm FT}^{(4)}(\mu) \;=\; \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \, Z_{\mathcal{X}}^{(\rm base)} \, Z_{\rm KK} \, \mathcal{G}(\mu, t)$$

Why can we now expect a statement on power-law running?

It helps to think of this from an equivalent extra-dimensional field theory view point -
To see how RG running *emerges* we can insert an energy scale μ by putting a cut-off function $\mathscr{G}(\mu, t)$ into the one-loop Schwinger integral (c.f. Polchinski/Wetterich Exact RG):

e.g. $\mathcal{G}(\mu, t) \;=\; \Theta(\mu^{-2} - t) \Longrightarrow$

$$\langle \mathcal{X} \rangle^{(4)}_{\rm FT}(\mu) \;=\; \int_{\Lambda^{-2}}^{\mu^{-2}} \frac{dt}{t}\, Z^{(\rm base)}_{\mathcal{X}}\, Z_{\rm KK}$$

Why can we now expect a statement on power-law running?

It helps to think of this from an equivalent extra-dimensional field theory view point -
To see how RG running *emerges* we can insert an energy scale μ by putting a cut-off function $\mathcal{G}(\mu, t)$ into the one-loop Schwinger integral (c.f. Polchinski/Wetterich Exact RG):

$$\text{e.g. } \mathcal{G}(\mu, t) = \Theta(\mu^{-2} - t) \Longrightarrow$$

$$\langle \mathcal{X} \rangle^{(4)}_{\mathrm{FT}}(\mu) = \int_{\Lambda^{-2}}^{\mu^{-2}} \frac{dt}{t} Z^{(\mathrm{base})}_{\mathcal{X}} Z_{\mathrm{KK}}$$

In analogy with the string theory results we can always write

$$\begin{aligned} Z^{(\mathrm{base})}_{\mathcal{X}} &= \frac{1}{t^2}(C'_0 + C'_1 t + C'_2 t^2 + \ldots) \\ Z^{(\mathrm{base})}_{\mathcal{X}} \cdot Z_{\mathrm{KK}} &= \frac{1}{t^2}(C_0 + C_1 t + C_2 t^2 + \ldots) \end{aligned}$$

So at energies far below the KK scale, $\mu \ll 1/R$, we can set $Z_{\mathrm{KK}} = 1$ and we get 4D running. e.g. the gauge coupling running is given by the log divergent term, C_2', and we get

$$\Delta_G(\mu) \equiv \quad \langle \mathcal{X} \rangle^{(4)}_{\mathrm{FT}}(\mu) \;=\; C_2' \log(\Lambda^2/\mu^2) + \mathrm{const}$$

So at energies far below the KK scale, $\mu \ll 1/R$, we can set $Z_{\rm KK} = 1$ and we get 4D running. e.g. the gauge coupling running is given by the log divergent term, C_2', and we get

$$\Delta_G(\mu) \equiv \quad \langle \mathcal{X} \rangle^{(4)}_{\rm FT}(\mu) \;=\; C_2' \log(\Lambda^2/\mu^2) + {\rm const}$$

But at energies high above the KK scale, $\mu \gg 1/R$, the factor $Z_{\rm KK}$ resums and we get…

$$\begin{aligned} \langle \mathcal{X} \rangle^{(4)}_{\rm FT}(\mu) \;&=\; V_\delta \int_{\Lambda^{-2}}^{\mu^{-2}} \frac{dt}{t^{1+\delta/2}}\, Z_{\mathcal{X}}^{(\rm base)} \\ &=\; \frac{2}{\delta} \; V_\delta \quad C_2' \, (\Lambda^\delta - \mu^\delta) \end{aligned}$$

This μ-dependence is our power-law running while Λ^δ is absorbed into the RG scheme.

The crux of the matter: in modular invariant theory: $C_2' = 0$ ***if*** $\delta > 2$ ***!***

In other words there can be no $\delta > 2$ power law running, and moreover there is no contribution to *any* running (even logarithmic) from the states in the theory associated with $\delta > 2$ decompactification limits.

- The case of $\delta = 2$ is more subtle: these *can* give logarithmic running below the KK scale.
- However it is easy to see that however we cut-off the integral *there can be no* $\delta = 2$ *power-law running if there is no* $\delta > 2$ *running (which as we just saw is unphysical).*

Let's see an example: running in a theory with a $\delta = 2$ decompactification limit

Modular invariant renormalisation:

- SAA, Dienes, 2021

The main difference to the extra dimensional FT picture is that to maintain modular invariance, the cut-off function $\mathcal{G}(\mu, \tau)$ must itself be *modular invariant*

Let's see an example: running in a theory with a $\delta = 2$ decompactification limit

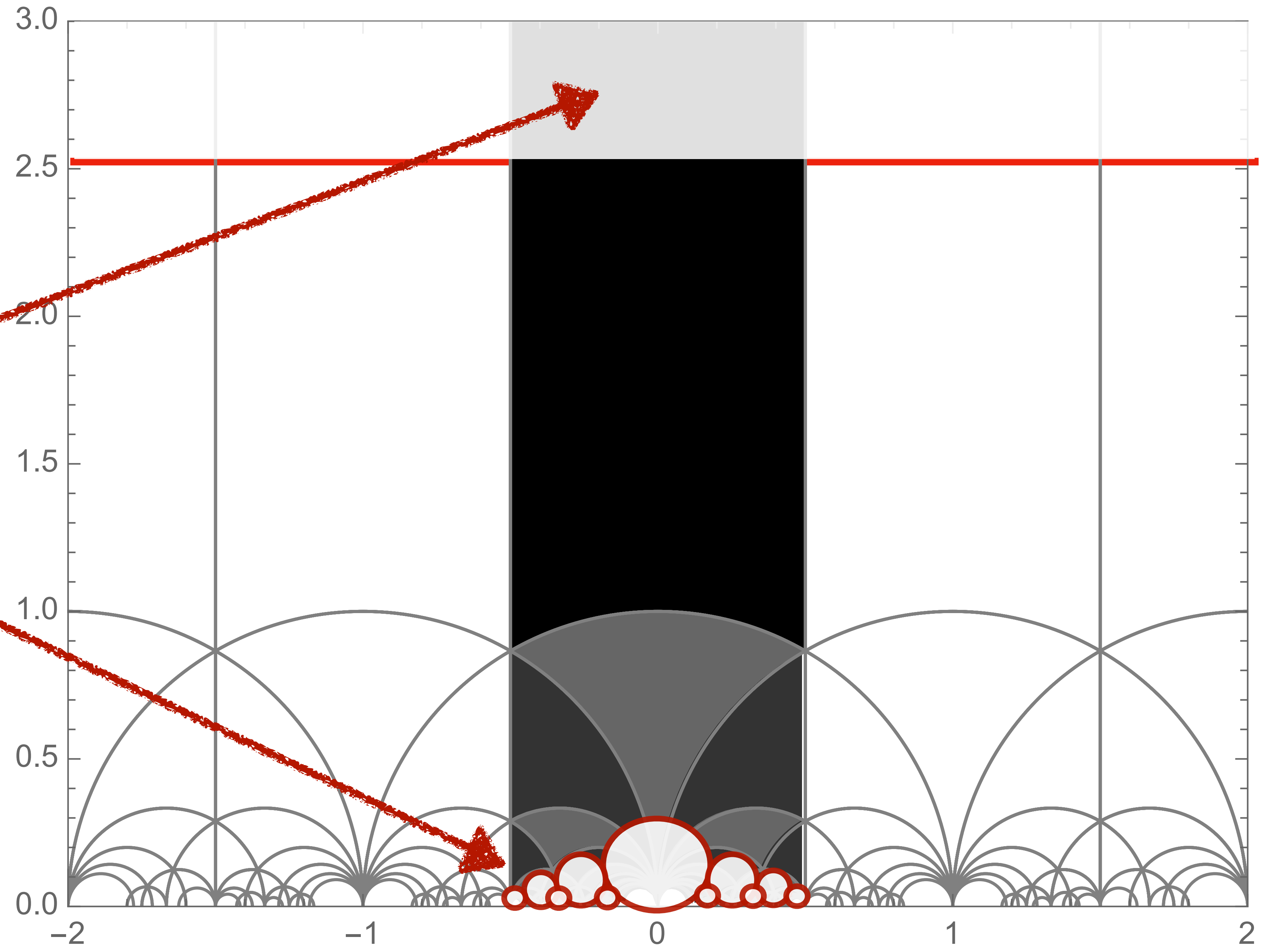
Modular invariant renormalisation:

- SAA, Dienes, 2021

The main difference to the extra dimensional FT picture is that to maintain modular invariance, the cut-off function $\mathscr{G}(\mu, \tau)$ must itself be *modular invariant* $\Longrightarrow$

$$\tau_* \equiv M_s^2/\mu^2$$

$\mathscr{G}$ suppresses integral in a modular invariant way around all the cusps



Let's see an example: running in a theory with a $\delta = 2$ decompactification limit

Modular invariant renormalisation:

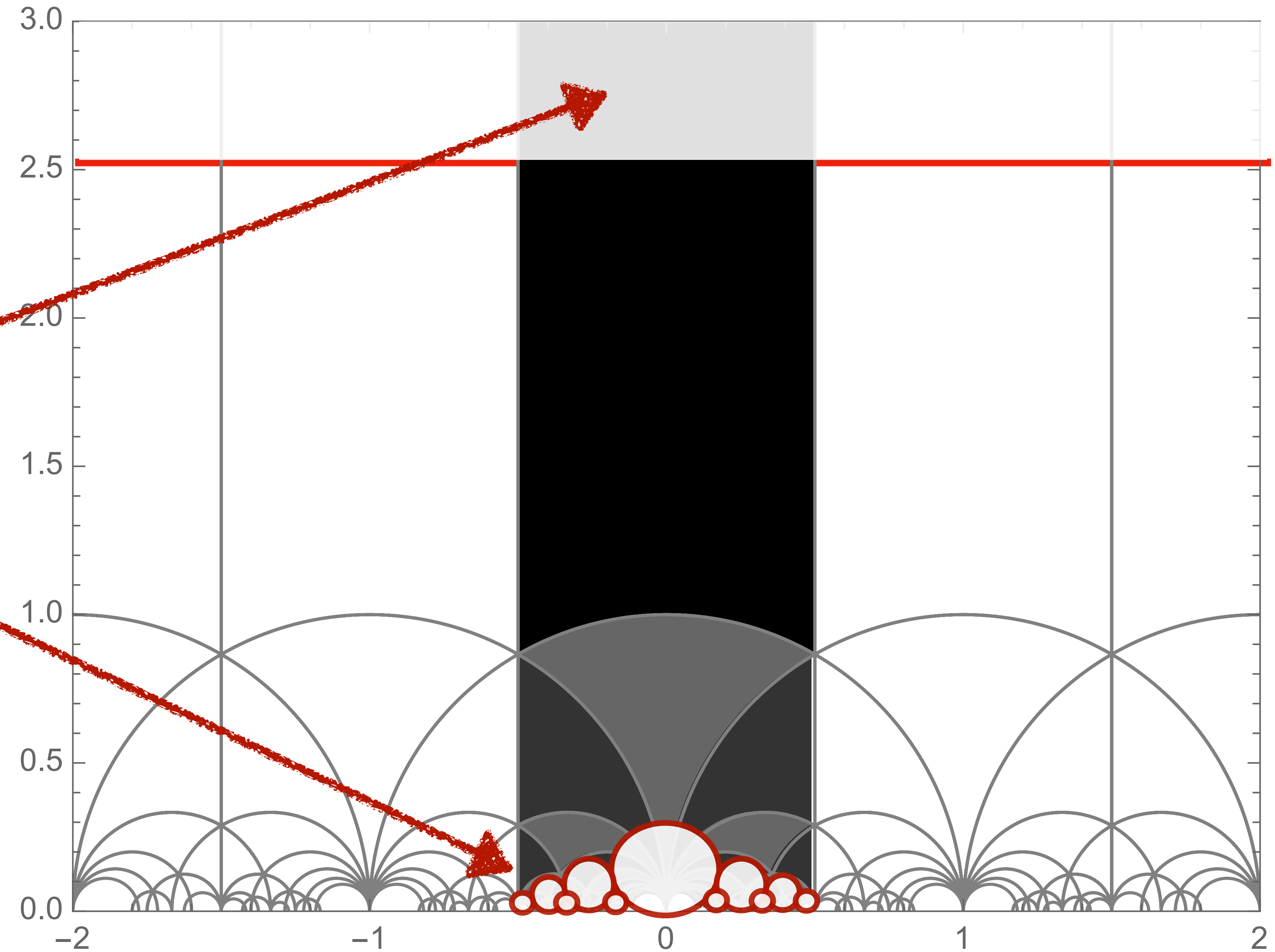
- SAA, Dienes, 2021

The main difference to the extra dimensional FT picture is that to maintain modular invariance, the cut-off function $\mathcal{G}(\mu, \tau)$ must itself be *modular invariant* $\Longrightarrow$

$$\tau_* \equiv M_s^2/\mu^2$$

NOTE: Consistency requires $\mathcal{G}$ invariant under

$$\mu \to M_s^2/\mu$$



Inserting such a regulator cut-off function with a 2-torus volume factor we can compare with the famous result of Dixon, Kaplunovsky and Louis, but recovering the entire energy dependence in Bessel functions …

SAA, Dienes, Nutricati

$$\Delta_G = \frac{-1}{1+a^2\rho}\Bigg\{ \log(cT_2U_2|\eta(T)\eta(U)|^4) + 2\log\sqrt{\rho}a + \frac{8}{\rho-1}\sum_{\gamma,\gamma'\in\Gamma_\infty\backslash\Gamma}\left[\tilde{\mathcal{K}}_0^{(0,1)}\left(\frac{2\pi}{a\sqrt{\gamma\cdot T_2\gamma'\cdot U_2}}\right) - \frac{1}{\rho}\tilde{\mathcal{K}}_1^{(1,2)}\left(\frac{2\pi}{a\sqrt{\gamma\cdot T_2\gamma'\cdot U_2}}\right)\right]\Bigg\},$$

where

$$\tilde{\mathcal{K}}_\nu^{(n,p)}(z,\rho) = \sum_{k,r=1}^{\infty}(krz)^n\Big(K_\nu(krz/\rho) - \rho^p K_\nu(krz)\Big)$$

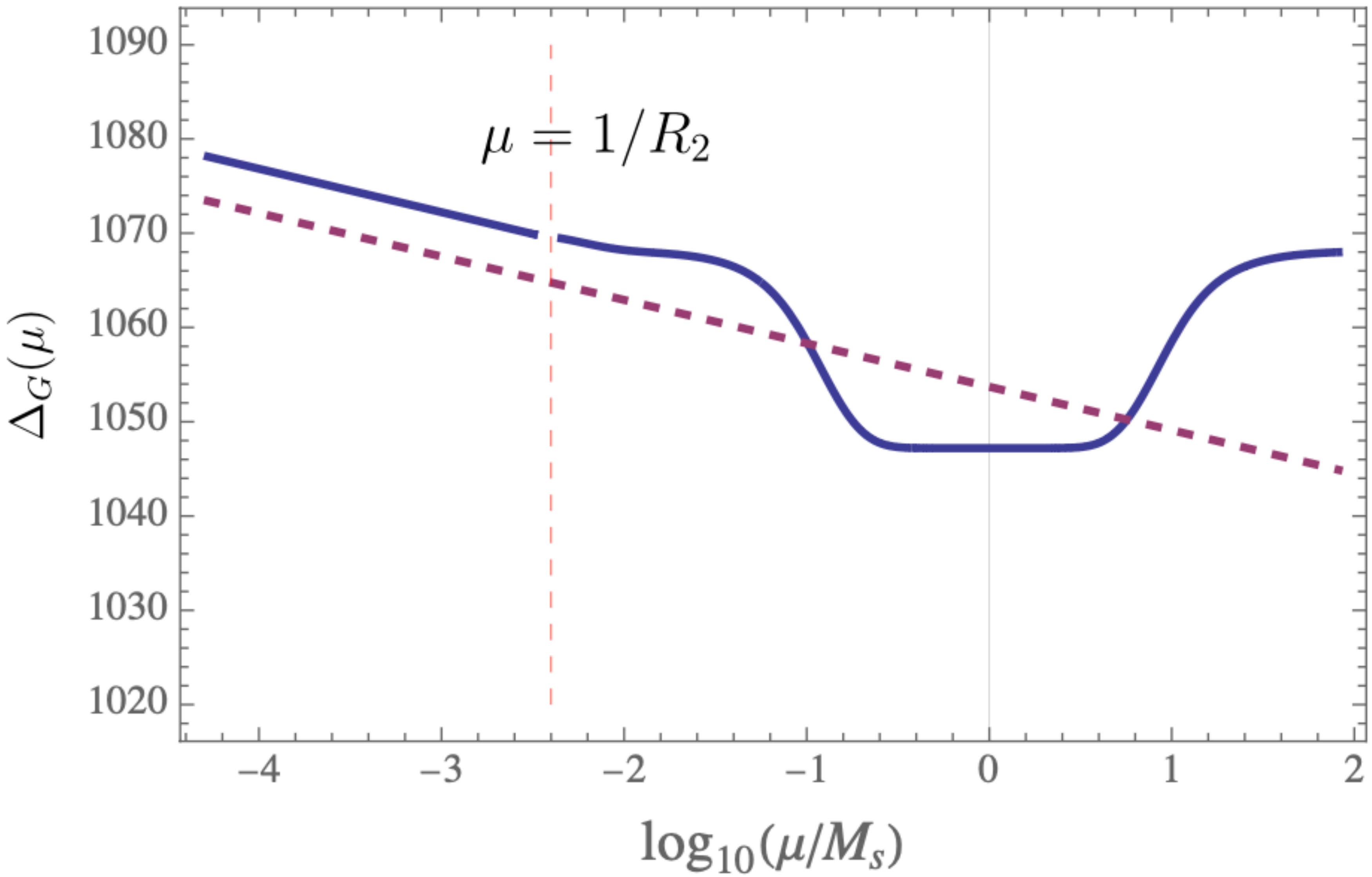
$$c(\rho) \equiv 16\pi^2\rho^{-\frac{\rho+1}{\rho-1}}e^{-2(\gamma+1)}$$

$$\mu = \sqrt{\rho}aM_s$$

Inserting such a regulator cut-off function with a 2-torus volume factor we can compare with the famous result of Dixon, Kaplunovsky and Louis, but recovering the entire energy dependence in Bessel functions …

SAA, Dienes, Nutricati

Rectangular torus:
$T_2 = R_2 R_1 = 1000$
$U_2 = R_2 / R_1 = 20$



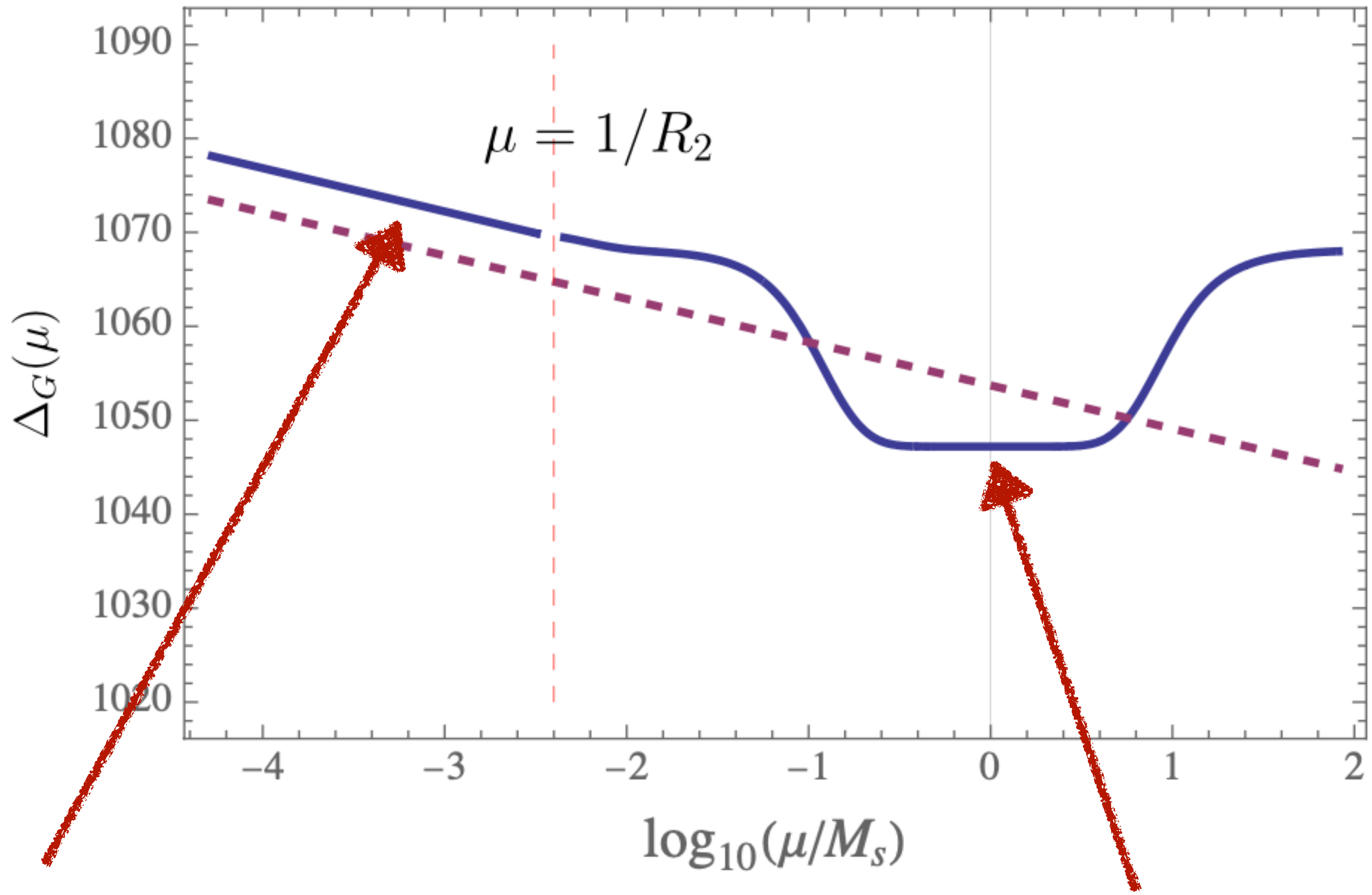
Inserting such a regulator cut-off function with a 2-torus volume factor we can compare with the famous result of Dixon, Kaplunovsky and Louis, but recovering the entire energy dependence in Bessel functions …

SAA, Dienes, Nutricati

Rectangular torus:

$T_2 = R_2 R_1 = 1000$

$U_2 = R_2/R_1 = 20$



$$\Delta_G \approx \frac{\pi}{3}\left(M_s^2 R_1 R_2 + \frac{R_2}{R_1}\right) - 2\log\left(\mu R_2\right)$$

$$\Delta_G \approx \frac{\pi}{3} M_s^2 R_1 R_2$$

Conclusions

- Important role of supertraces in allowing an EFT emerge from any modular invariant theory.
- RS provides completely model agnostic understanding of this process
- Each decompactification limit lead to a set of supertrace constraints
- A form of non-renormalisation theorem which is satisfied due to modular invariance
- Phenomenological consequences - no power law running
- Removes "technical hierarchies": i.e. all the heavy modes yield a constant piece that may be large but which is separated from light modes.
- Links/solutions to hierarchy problem?