Fluxes, Tadpoles and Singularities

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F-Theory

gives us a huge set of 4D $\mathcal{N} = 1$ string vacua.
Data:

- a Calabi-Yau fourfold X with an elliptic fibration $E \to X \to B$
- a flux G_4 s.t. (restricting to 4D SUSY Minkowski)

$$G_4 \wedge J = 0 \qquad\qquad G_4 + \frac{c_2(X)}{2} \in H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

'primitive Hodge cycle'

$$\tfrac{1}{2} \int_X G_4 \wedge G_4 = \frac{\chi(X)}{24} + N_{D3} \qquad\qquad \text{(tadpole)}$$

0th order approximation of effective $4D$ physics:

- number of complex structure moduli fixed by G_4
- non-abelian gauge groups from reducible fibres over codim$_{\mathbb{C}} = 1$ loci of B

Tadpoles

common lore: a generic choice of G_4 will already fix all complex structure moduli
however, G_4 is quantized ...

- ... in which sense can something discrete be generic?
- ... especially if its length is bounded?

There is some tension here: **'tadpole problem'** [Bena, Blaback, Grana, Lust]

General Hodge Cycles

For a given Hodge cycle G_4, the locus in $CS(X)$ where it is of Hodge type $(2, 2)$ is called its **Hodge locus**. If this is just points, G_4 is a **general Hodge cycle**.

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Which points $p \in CS(X)$ are selected by the condition that all complex structure moduli are stabilized by fluxes?

Exciting question in F-Theory, spectrum is very sensitive to this!

Finiteness results (even for $W \neq 0$) established by [B.Bakker, T.Grimm, C.Schnell, J.Tsimerman].

A toy model: $K3 \times K3$

Let $X = S_1 \times S_2$. We have $c_2(X) =$ even and $\chi(X) = 24^2$. All complex structure moduli are stablized provided that we can find a γ s.t.

$$G_4 = Re\left(\gamma \Omega_1^{2,0} \wedge \bar{\Omega}_2^{2,0}\right) \in H^{2,2}(X)$$

is integral. [P.Aspinwall, R.Kallosh]. This implies that both S_1 and S_2 are singular ('attractive'), i.e. the Picard group

$$Pic(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$$

has rank $pic(S) = 20$.

This implies no complex structure deformations are left to deform by.

Attractive K3s

Theorem [T.Shioda, H.Inose]: K3 surfaces S with $pic(S) = 20$ are classified by inner form of transcendental lattice $T_S = Pic^{\perp} \subset H^2(S,\mathbb{Z})$:

$$Q = \begin{pmatrix} p^2 & p\cdot q \\ p\cdot q & q^2 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = [a,b,c]$$

mod $SL(2,\mathbb{Z})$, where $\Omega^{2,0} = p + iq$ and $p, q \in H^2(S,\mathbb{Z})$.

There exists a γ such that

$$G_4 = Re\left(\gamma\Omega_1^{2,0} \wedge \bar{\Omega}_2^{2,0}\right) \in H^{2,2}(X)$$

is integral if and only if $det(Q_1 Q_2) = D_1 D_2$ is a perfect square.

Solutions in M and F-Theory

This gives a finite list of solutions using $\int_{S_1 \times S_2} G_4 \wedge G_4 \leq 48$ [P.Aspinwall, R.Kallosh; AB, Y.Kimura, T.Watari].

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A primitive embedding $f : U \to Pic(S)$ fixes an elliptic fibration.

$$W = U^{\perp} \in Pic(S)$$

and we can write

$Pic(S) = U \oplus W$ W = 'frame lattice' = reducible fibres + Mordell-Weil

$$W_{root} = \langle \{\eta \in W | \eta^2 = -2\} \rangle = \oplus_i \Gamma_i$$

determines non-abelian gauge algebra.

Kneser-Nishiyama

All W can be found as

$$W(-1) = T_0^{\perp} \in N_i \, ,$$

where

- $T_0 = T_X^{\perp} \subset E_8$
- N_i ($i = 1..24$) are the Niemeier lattices (even, self-dual 24 dim).

i.e. one needs to work out all primitive embeddings of T_0 into the Niemeier lattices.

e.g. for $T_X = A_2$, $T_0 = E_6$ which we can embedd into

$$\begin{aligned}
N_\beta &= D_{16} \oplus E_8; \mathbb{Z}_2 \\
N_\gamma &= E_8^3 \\
N_\zeta &= A_{17} \oplus E_7; \mathbb{Z}_6 \\
N_\eta &= D_{10} \oplus E_7^2; \mathbb{Z}_2^2 \\
N_\lambda &= A_{11} \oplus D_7 \oplus E_6; \mathbb{Z}_4 \times \mathbb{Z}_3 \\
N_\mu &= E_6^4; \mathbb{Z}_3^2
\end{aligned}$$

All F-Theory solutions

The complete set of attractive K3 surfaces appearing is [AB, Y.Kimura, T.Watari]

$$
\begin{aligned}
Q =&[a, b, c] \in \\
&\{[1,0,1],[1,1,1],[2,0,1],[2,1,1],[3,0,1],[3,1,1],[4,0,1],[4,1,1],[5,0,1],[5,1,1],[6,0,1],\\
&[6,1,1],[2,0,2],[2,1,2],[2,2,2],[3,0,2],[3,1,2],[3,2,2],[4,0,2],[4,2,2],[6,0,2],[6,2,2],\\
&[3,0,3],[3,3,3],[6,0,3],[6,3,3],[4,0,4],[4,4,4],[5,0,5],[5,5,5],[6,0,6],[6,6,6],[7,7,7],\\
&[8,8,8]\}
\end{aligned}
$$

[Braun, Fraiman, Grana, Lust, Parra De Freitas]: W_{root} **is non-zero for all elliptic fibrations on any K3 surface in the above list** $\rightarrow$ **we always get non-abelian gauge groups!**

There are attractive K3 surfaces with elliptic fibrations with $W_{root} = \emptyset$, but these do not appear due to the tadpole bound. The first example appears for $\frac{1}{2}G \cdot G = 30$ and is $[10, 10, 10]$. Here T_0 embedds into the Leech lattice (which has no roots).

Remarkably, the points in moduli space selected by

- G_4 is a general Hodge cycle
- the tadpole bound

are very special indeed. Dropping either of these conditions non-Abelian gauge groups are not forced on us.

What about honest 4folds?

Most well-studied example: the Fermat sextic 4fold X in $\mathbb{CP}^5$ (no elliptic fibration so we are doing M-Theory) see also [K.Becker,E.Gonzalo, J.Walcher, T.Wrase]

$$0 = Q(x_i) = \sum_{i=0}^{5} x_i^6$$

$$h^{3,1}(X) = 426 \quad h^{2,2}_{prim}(X) = 1751 \quad \chi(X) = 2610$$

middle cohomology given by residues

$$H^{4-k,k}(X) = \langle \omega_\beta \rangle \qquad \sum_i \beta_i = 6k$$

$$\omega_\beta := Res\left(\frac{x^\beta \Omega_0}{Q(x)^{k+1}}\right)$$

and Hodge locus has codimension

$$\text{rk}\, \rho_{IJ}(G) := \text{rk}\, \omega_{\beta_I+\beta_J} \cdot G \qquad \sum_i (\beta_I)_i = 6$$

algebraic cycles

For any factorization

$$Q = f_0 P_0 + f_1 P_1 + f_2 P_2$$

there is an algebraic cycle C (of complete intersection type) defined by

$$C : \{f_0 = f_1 = f_2 = 0\}$$

The Poincaré dual is of type $(2, 2)$ and integral (but not primitive). All we need to know about these is known.

e.g.

(d_0, d_1, d_2)	$C^2_{f_0 f_1 f_2}$	rk $\rho_{IJ}(C_{f_0 f_1 f_2})$
$(1,1,1)$	21	19
$(1,1,2)$	34	32
$(1,1,3)$	39	37
$(1,2,2)$	56	54
$(1,2,3)$	66	62
$(1,3,3)$	81	71
$(2,2,2)$	96	92
$(2,2,3)$	120	106
$(2,3,3)$	162	122
$(3,3,3)$	243	141

A naive count would imply you wildly overshoot the tadpole bound of $435/4$.

Can implement correct quantization condition and primitivity [AB, R. Valandro; AB, H. Fortin, D. Lopez Garcia, R. Villaflor]. Record shortest general Hodge cycle has

$$\tfrac{1}{2} G \cdot G = \frac{567}{4} > \frac{435}{4} \, .$$

and remarkably

$$\frac{567}{4} / 426 \simeq 1/3$$

as originally conjectured by [Bena, Blaback, Grana, Lust].

However:

- Not exhaustive search seach ($h^{2,2}_{prim} = 1751$)
- There exists no proof of the Hodge conjecture for X.

residues

It is known that

$$H^4(X,\mathbb{Z}) = \langle \delta_\beta, C^\ell_\sigma \rangle_\mathbb{Z}$$

$$\int_{C^\ell_\sigma} \omega_\beta = \begin{cases} (2\pi i)^2 \frac{\mathrm{sgn}(\sigma)}{6^3 \cdot 2} \mu^{\sum_{e=0}^{2} (\beta_{\sigma(2e)}+1)(2(\ell_{2e+1}-\ell_{2e})+1)} & \text{if } \beta_{\sigma(2e-2)} + \beta_{\sigma(2e-1)} = 4 \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{\delta_{\beta'}} \omega_\beta = \frac{(-1)^{|\beta|}}{6^5} \frac{1}{|\beta|! 2\pi i} \prod_{i=0}^{5} \Gamma\left(\frac{\beta_i+1}{6}\right) \left(\zeta^{(\beta'_i+1)(\beta_i+1)} - \zeta^{(\beta'_i)(\beta_i+1)}\right)$$

so that we can work out a basis of $H^4(X,\mathbb{Z})$ in terms of the residues ω_β and compute ρ_{IJ} for any choice of G_4.

This is again very hard due to the high dimensionality of the problem. We worked this out in [AB, H. Fortin, D. Lopez Garcia, R. Villaflor] by restricting to forms invariant under discrete symmetries, very quickly gets too expensive computationally.

There is a pretty vast landscape of computations one might want to do and conjectures one might want to make ...

1. High dimensionality makes brute force attacks fairly unfeasable.
2. No analogues of the details we know for the Fermat sextic have been developed (but this seems doable).
3. A better understanding will almost certainly involve arithmetic (see e.g. work by [O.DeWolfe, A.Giryavets, S.Kachru, W.Taylor; K.Kanno, T.Watari; S.Kachru, R.Nally, W.Yang; H.Jockers, S.Kotlewski, P.Kuusela; P.Candelas, X.de la Ossa, P.Kuusela, J.McGovern; T.Grimm, D.van de Heisteeg]).
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Thank you!